AP · Advanced Placement · College Board

Chuyên luyện thi IELTS/SAT → AP / IB / A-levels

8020 ENGLISH

AP Calculus BC

Giáo trình luyện thi chuyên sâu

Song ngữ Anh–Việt

8020 ENGLISH

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Lời mở đầu · Foreword

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9.0 IELTS cao nhất	**1570** SAT cao nhất
5/5 AP — điểm tuyệt đối	**90/100** IB Diploma
100% GV $\geq$ 8.0 IELTS / 1500 SAT	**CMU** Tiến sĩ ĐH #1 Hoa Kỳ về CS

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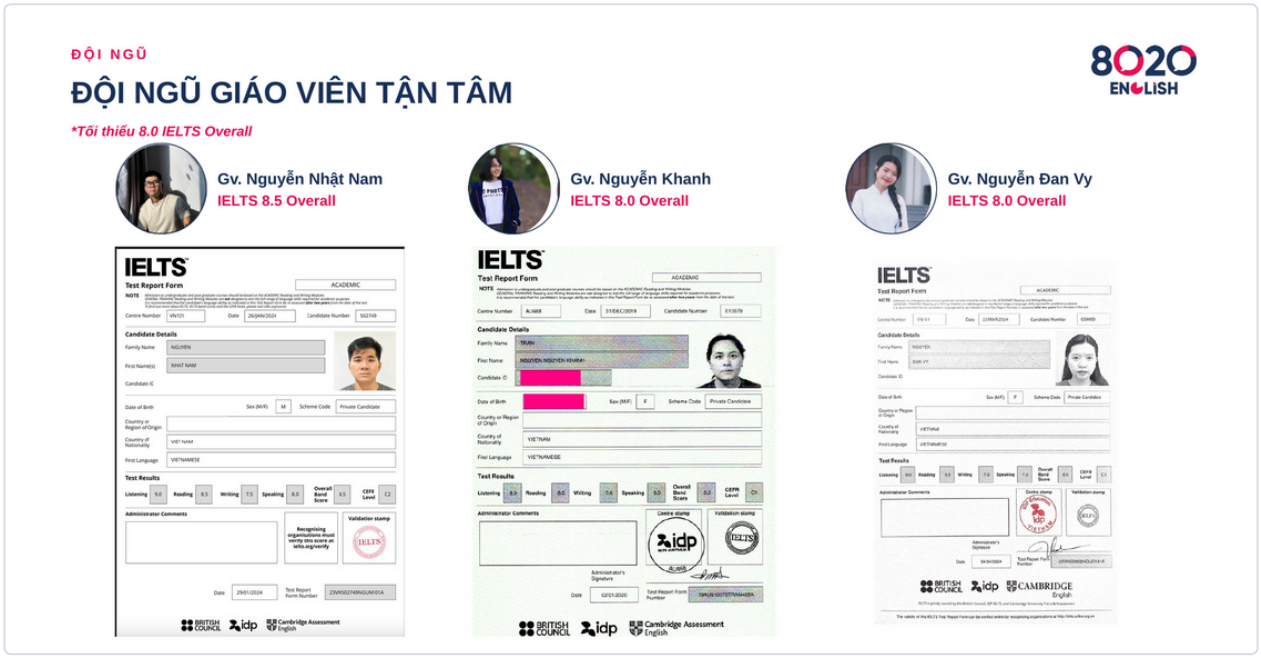
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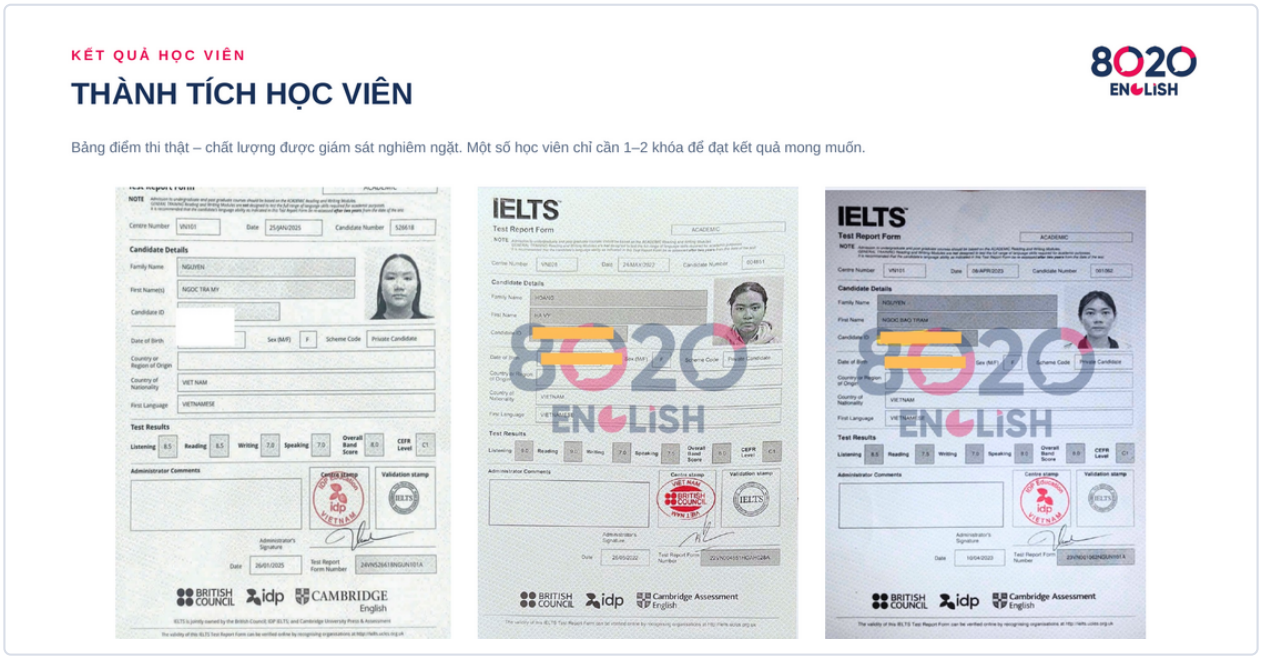
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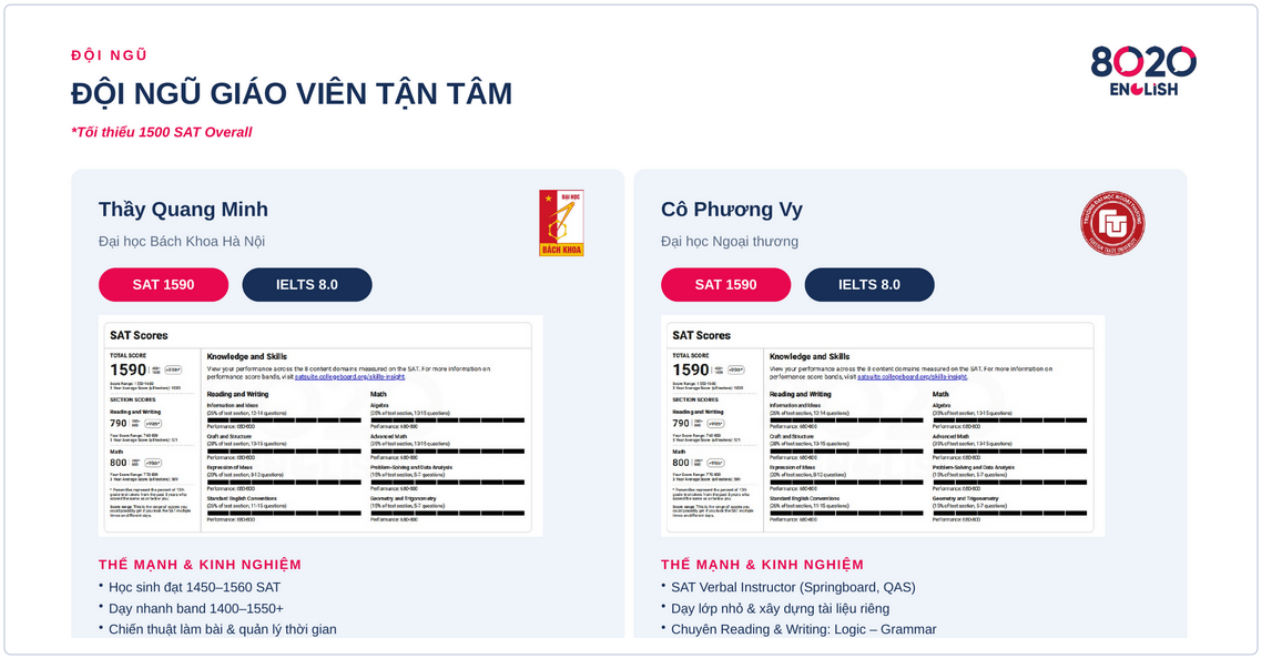
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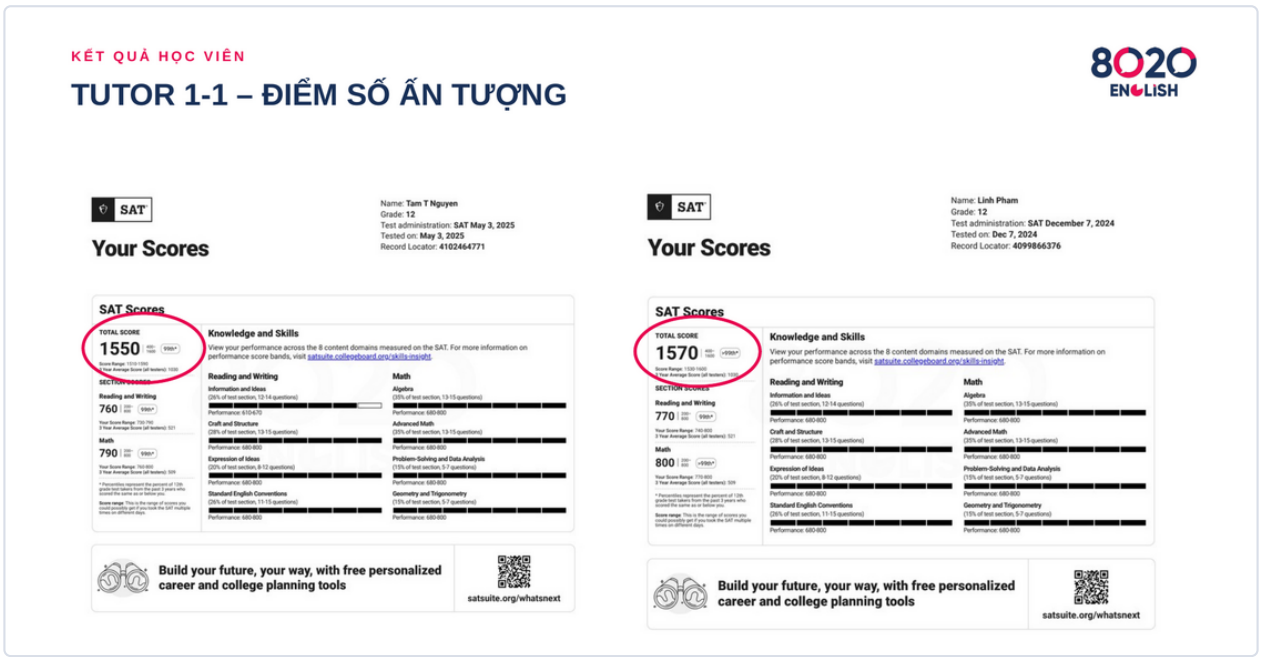
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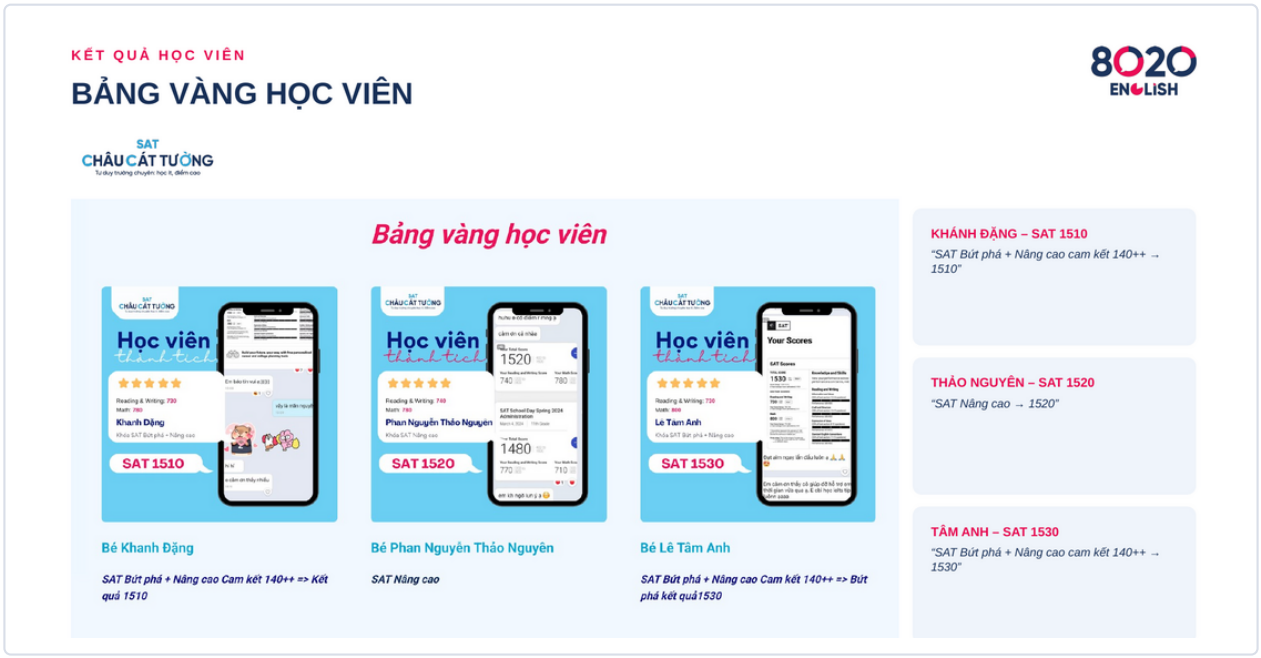
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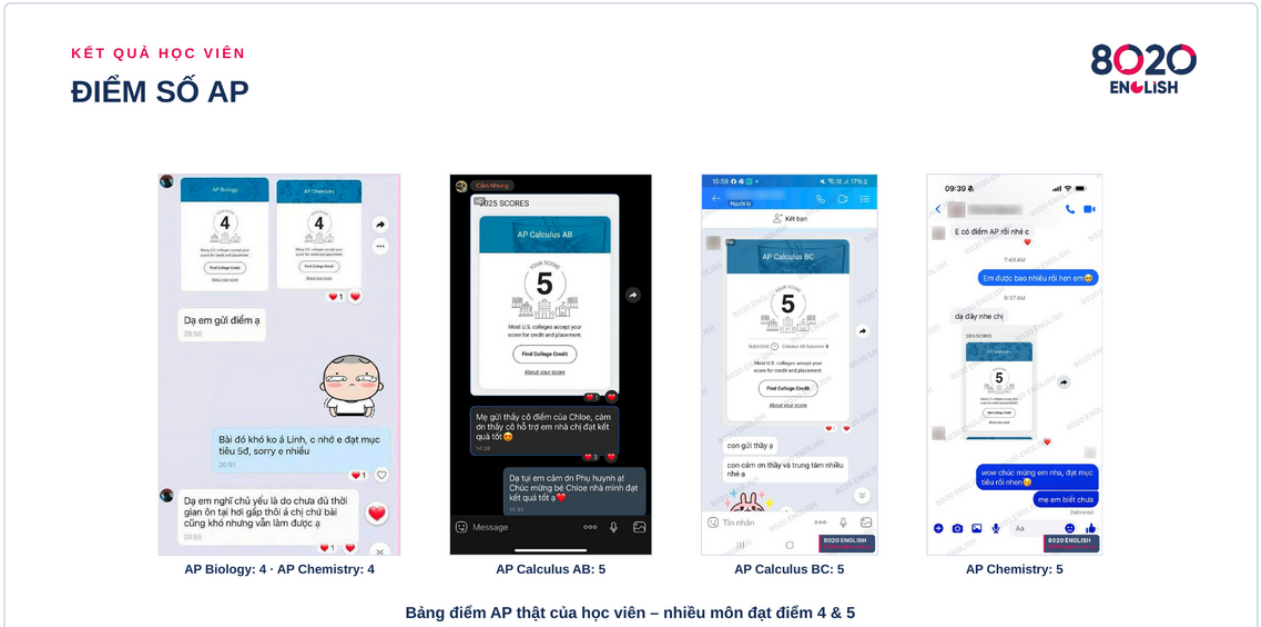
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Điểm AP thật — Calculus AB/BC 5/5 · Biology & Chemistry 4–5 (College Board)

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Limits and Continuity

Mục tiêu Unit: Hiểu khái niệm giới hạn (limit) một cách trực quan và qua kí hiệu hình thức; ước lượng giới hạn từ đồ thị và bảng giá trị; tính giới hạn bằng các định lý giới hạn (limit laws) và các kỹ thuật đại số (phân tích nhân tử, nhân liên hợp, quy đồng mẫu, xét hàm từng khúc); nắm hai giới hạn lượng giác đặc biệt và Định lý Kẹp (Squeeze Theorem); phân loại các dạng gián đoạn (khử được, nhảy, vô cực) và phát biểu định nghĩa liên tục tại một điểm cũng như trên một khoảng; xử lý giới hạn vô cực, tiệm cận đứng, tiệm cận ngang (kể cả với hàm mũ); và áp dụng Định lý Giá trị Trung gian (Intermediate Value Theorem) để chứng minh sự tồn tại của nghiệm hoặc của một giá trị hàm số.

1.1 From Average Rate of Change to the Idea of a Limit

Calculus begins with a single question that algebra cannot answer on its own: how fast is a quantity changing *at a single instant*? Algebra is very good at describing change over an **interval** — given two data points, it readily produces a single average rate — but a single instant is not an interval at all, so no ordinary algebraic ratio can be built from it directly. Every major idea in the chapters ahead, from the derivative (an instantaneous rate of change) to the definite integral (an instantaneous accumulation, built from infinitely many infinitesimally thin pieces), is constructed by taking exactly this kind of limiting process and making it precise; understanding limits thoroughly here is what makes those later constructions possible. If a car's position at time t (in seconds) is $s(t) = t^2 + 2t$ (in meters), the **average rate of change** of s over an interval $[a, b]$ is the slope of the secant line joining $(a, s(a))$ and $(b, s(b))$:

$$\text{average rate of change on } [a,b] = \frac{s(b) - s(a)}{b - a}.$$

This number is easy to compute and tells us the car's average velocity over the whole interval, but it says nothing about how fast the car was moving at one particular moment, say $t = 1$. To isolate the behavior at a single instant, we shrink the interval down: instead of $[1, 4]$, consider $[1, 1 + h]$ for smaller and smaller values of h, and ask what the average rate of change over $[1, 1 + h]$ approaches as h shrinks toward 0 (without ever letting h actually equal 0, since the interval $[1, 1]$ has zero width and the ratio $\frac{s(1+h)-s(1)}{h}$ is undefined at $h = 0$). The value this ratio *approaches* — not the value it takes on at $h = 0$, which does not even exist — is called a **limit**, and it is the central object of study of this entire chapter.

Ví dụ mẫu · Average vs. instantaneous velocity

A car's position is $s(t) = t^2 + 2t$ meters, t in seconds. (a) Find the average velocity on $[1, 4]$. (b) Estimate the instantaneous velocity at $t = 1$ by computing the average velocity on $[1, 1 + h]$ and examining what happens as $h \to 0$.

(a) $s(4) = 16 + 8 = 24$ and $s(1) = 1 + 2 = 3$, so

$$\text{average velocity} = \frac{s(4) - s(1)}{4 - 1} = \frac{24 - 3}{3} = \frac{21}{3} = 7 \text{ m/s}.$$

(b) Compute the difference quotient on $[1, 1 + h]$:

$$s(1 + h) = (1 + h)^2 + 2(1 + h) = 1 + 2h + h^2 + 2 + 2h = 3 + 4h + h^2,$$

so

$$\frac{s(1+h)-s(1)}{h} = \frac{(3+4h+h^2)-3}{h} = \frac{4h+h^2}{h} = 4+h \quad (h \neq 0).$$

As h is taken closer and closer to 0 (from either side), the expression $4+h$ gets closer and closer to 4. We say the limit of the average velocity, as $h \to 0$, is 4 — and this is exactly what we mean by the **instantaneous velocity** at $t = 1$: 4 m/s. Notice that the average velocity over the much longer interval $[1, 4]$, at 7 m/s, is a completely different number from the instantaneous velocity at the single instant $t = 1$.

VN **GIẢI THÍCH TIẾNG VIỆT**

Tốc độ thay đổi trung bình (average rate of change) trên một đoạn $[a, b]$ chỉ là độ dốc của đường thẳng nối hai điểm đầu-cuối — tính được ngay bằng đại số. Nhưng tốc độ thay đổi **tức thời** (instantaneous) tại một điểm duy nhất đòi hỏi ta thu hẹp đoạn $[a, b]$ lại thành một đoạn cực ngắn quanh điểm đó rồi xem tỉ số thay đổi **tiến tới** giá trị nào — đây chính là khái niệm **giới hạn (limit)**. Đây là ý tưởng nền tảng sẽ trở lại trong Chương sau khi ta định nghĩa đạo hàm (derivative) chính là giới hạn này.

1.2 Defining Limits and One-Sided Limits

1.2.1 The intuitive meaning of a limit

We write

$$\lim_{x \to c} f(x) = L$$

to mean: as x takes values closer and closer to c (from either side, but always with $x \neq c$), the corresponding outputs $f(x)$ get closer and closer to the single number L. Read this as "the limit of $f(x)$, as x approaches c, is L."

Approach, not arrival

A limit describes what $f(x)$ is **heading toward** as x approaches c — it makes no claim whatsoever about the actual value $f(c)$. Three situations can occur, and all three are perfectly compatible with $\lim_{x \to c} f(x) = L$: $f(c) = L$ (the ordinary case); $f(c)$ equals some other number entirely; or $f(c)$ is not defined at all. The limit only cares about values of x **near** c, never about $x = c$ itself.

VN **GIẢI THÍCH TIẾNG VIỆT**

Giới hạn mô tả giá trị mà $f(x)$ **tiến tới** khi x tiến rất gần c — hoàn toàn không liên quan đến giá trị thật của $f(c)$. Có ba khả năng đều tương thích với $\lim_{x \to c} f(x) = L$: $f(c) = L$ (trường hợp thường gặp); $f(c)$ bằng một số khác; hoặc $f(c)$ không xác định. Học sinh mới học thường nhầm lẫn "tính giới hạn" với "tính giá trị hàm số" — đây là hai việc khác nhau.

1.2.2 One-sided limits

Sometimes it matters from which side x approaches c. The **left-hand limit**, written $\lim_{x \to c^-} f(x)$, only considers values of x less than c approaching c; the **right-hand limit**, written $\lim_{x \to c^+} f(x)$, only considers values of x greater than c approaching c. The two-sided limit exists precisely when both one-sided limits exist and agree.

$$\lim_{x\to c} f(x) = L \iff \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L.$$

If the two one-sided limits are unequal, or if either one fails to exist (e.g., is infinite), the two-sided limit $\lim_{x\to c} f(x)$ **does not exist** (abbreviated DNE).

VN **GIẢI THÍCH TIẾNG VIỆT**

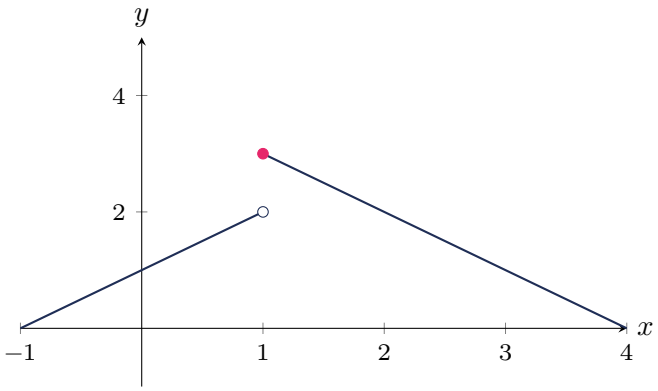
Giới hạn trái $\lim_{x\to c^-} f(x)$ chỉ xét x tiến tới c từ phía **nhỏ hơn** c; giới hạn phải $\lim_{x\to c^+} f(x)$ chỉ xét x tiến tới c từ phía **lớn hơn** c. Giới hạn hai phía chỉ tồn tại khi và chỉ khi hai giới hạn một phía **cùng tồn tại và bằng nhau**. Đây là công cụ chủ đạo để xét giới hạn của hàm từng khúc (piecewise) và để xét gián đoạn kiểu "nhảy" (jump discontinuity) ở phần sau chương này.

(!) Lỗi thường gặp: "$f(c)$ không xác định" và "$\lim_{x\to c} f(x)$ không tồn tại" là hai điều **hoàn toàn khác nhau**, dù học sinh rất hay gộp chung. Một hàm số có thể không xác định tại c (ví dụ có một "lỗ hổng" trên đồ thị) nhưng giới hạn tại c vẫn tồn tại và hữu hạn — đây chính là bản chất của **gián đoạn khử được** (removable discontinuity), trình bày chi tiết ở phần sau. Ngược lại, $f(c)$ có thể xác định rõ ràng nhưng giới hạn tại c vẫn không tồn tại (ví dụ gián đoạn kiểu nhảy). Luôn kiểm tra hai câu hỏi này một cách độc lập.

1.3 Estimating Limits Graphically and Numerically

1.3.1 Reading limits from a graph

On a graph, $\lim_{x\to c} f(x)$ is the height the curve is heading toward as you trace it from the left and from the right toward $x = c$ — completely independent of whether the graph shows an open circle, a filled dot, or nothing at all plotted directly above $x = c$. To read a one-sided limit, trace the curve from only one direction.



$f(x) = \begin{cases} x+1, & x<1 \\ -x+4, & x\geq 1 \end{cases}$. Tracing from the left, the curve heads toward the open circle at height 2: $\lim_{x\to 1^-} f(x) = 2$. Tracing from the right, the curve heads toward the filled dot at height 3: $\lim_{x\to 1^+} f(x) = 3$. Since $2 \neq 3$, $\lim_{x\to 1} f(x)$ does not exist, even though $f(1) = 3$ is perfectly well defined.

VN **GIẢI THÍCH TIẾNG VIỆT**

Trên đồ thị, giới hạn tại $x = c$ chính là **độ cao mà đường cong đang hướng tới** khi ta "đi dọc" theo đường cong từ trái và từ phải về phía $x = c$ — hoàn toàn không quan tâm điểm đó vẽ bằng vòng tròn rỗng, chấm đặc, hay không vẽ gì cả. Trong hình trên, giới hạn trái là 2 (vòng tròn rỗng), giới hạn phải là 3 (chấm đặc, cũng chính là $f(1)$); vì hai giới hạn một phía khác nhau nên giới hạn hai phía **không tồn tại**.

1.3.2 Estimating limits from a table of values

Another way to estimate a limit is to build a table of $f(x)$ for values of x that get closer and closer to c from both sides, and observe the pattern the outputs follow. This numerical approach is especially useful when a formula is hard to simplify by hand, or as a way to check an algebraic answer.

Ví dụ mẫu · Estimating a limit numerically

Estimate $\lim_{x\to 1} \dfrac{x^2-1}{x-1}$ using a table of values.

x	0.9	0.99	0.999	1.001	1.01
$f(x)$	1.9	1.99	1.999	2.001	2.01

As x approaches 1 from either side, $f(x)$ appears to approach 2. This matches the algebraic computation: for $x \neq 1$, $\dfrac{x^2-1}{x-1} = \dfrac{(x-1)(x+1)}{x-1} = x+1$, and $x+1 \to 2$ as $x \to 1$. So $\lim_{x\to 1} \dfrac{x^2-1}{x-1} = 2$, even though the original expression is the indeterminate form $\frac{0}{0}$ at $x = 1$ itself.

VN GIẢI THÍCH TIẾNG VIỆT

Bảng giá trị là công cụ trực quan để **ước lượng** giới hạn: cho x tiến gần c từ cả hai phía bằng những giá trị cụ thể (ví dụ $0.9,\ 0.99,\ 0.999, \ldots$) và quan sát $f(x)$ tiến về đâu. Đây chỉ là phương pháp **ước lượng**, không phải chứng minh chặt chẽ — muốn khẳng định chắc chắn giá trị giới hạn, ta vẫn cần kỹ thuật đại số (Mục tiếp theo) hoặc phép thế trực tiếp.

1.4 Determining Limits Using Algebraic Properties of Limits

When both $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ exist (as finite numbers), limits interact predictably with the basic algebraic operations. These **limit laws** let us break a complicated limit into simpler pieces.

The limit laws

Suppose $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, and k is a constant. Then:

- **Sum:** $\lim_{x\to c} [f(x) + g(x)] = L + M$
- **Difference:** $\lim_{x\to c} [f(x) - g(x)] = L - M$
- **Constant multiple:** $\lim_{x\to c} [k\, f(x)] = kL$
- **Product:** $\lim_{x\to c} [f(x)g(x)] = LM$
- **Quotient:** $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, *provided* $M \neq 0$
- **Power:** $\lim_{x\to c} [f(x)]^n = L^n$ for any positive integer n (and more generally for rational powers $L^{p/q}$, whenever $L^{p/q}$ is defined)

A direct and enormously useful consequence of the limit laws is that for a function built entirely out of sums, differences, products, quotients, and powers of simpler continuous pieces — polynomials, root functions, sine and cosine, exponentials, logarithms — the limit at any point c inside the

function's domain can be found by **direct substitution**:

$$\lim_{x\to c} f(x) = f(c) \qquad \text{whenever } f \text{ is continuous at } c.$$

This is by far the fastest technique and should always be tried first. It fails only when substitution produces an undefined expression, such as division by zero.

Ví dụ mẫu · Applying the limit laws

Suppose $\lim_{x\to 5} p(x) = 4$ and $\lim_{x\to 5} q(x) = 13$. (a) Find $\lim_{x\to 5}\big[p(x)q(x)+5\big]$. (b) Find $\lim_{x\to 5}\sqrt{q(x)-p(x)}$. (c) Evaluate $\lim_{x\to 2}\dfrac{x^3+2x-1}{x^2+1}$ by direct substitution.

(a) By the product law, $\lim p(x)q(x) = (4)(13) = 52$; by the sum law, $\lim\big[p(x)q(x)+5\big] = 52+5 = 57$.

(b) By the difference law, $\lim\big[q(x)-p(x)\big] = 13-4 = 9$; by the power law (square root is the power $\frac{1}{2}$), $\lim\sqrt{q(x)-p(x)} = \sqrt{9} = 3$.

(c) $\dfrac{x^3+2x-1}{x^2+1}$ is a quotient of polynomials, continuous everywhere the denominator is nonzero; since $2^2+1 = 5 \neq 0$, substitute $x = 2$ directly:

$$\frac{2^3+2(2)-1}{2^2+1} = \frac{8+4-1}{4+1} = \frac{11}{5}.$$

VN **GIẢI THÍCH TIẾNG VIỆT**

Các **định lý giới hạn (limit laws)** cho phép "tách" một giới hạn phức tạp thành các giới hạn đơn giản hơn: giới hạn của tổng bằng tổng các giới hạn, giới hạn của tích bằng tích các giới hạn, v.v. — miễn là các giới hạn thành phần đều tồn tại (và với phép chia, mẫu số không tiến về 0). Hệ quả quan trọng nhất: nếu f **liên tục** tại c (đa thức, hàm căn, hàm lượng giác, hàm mũ/log trong miền xác định), ta chỉ cần **thế trực tiếp** $x = c$ vào — đây luôn là bước thử đầu tiên.

1.5 Determining Limits Using Algebraic Manipulation

Direct substitution fails whenever it produces an undefined expression — most commonly the **indeterminate form** $\frac{0}{0}$. This does not mean the limit is 0, and it does not mean the limit fails to exist; it means the expression must first be rewritten in an equivalent form for which substitution *does* work. The word "indeterminate" is precise: $\frac{0}{0}$ alone tells us nothing about the eventual value of the limit, since different functions that both produce $\frac{0}{0}$ at a point can have completely different limits there (or none at all) — the specific algebraic structure of the expression, not the symbol $\frac{0}{0}$ itself, determines the answer. Three algebraic techniques handle the overwhelming majority of such limits.

1.5.1 Factoring and canceling common factors

If both numerator and denominator share a common factor that vanishes at $x = c$, factor both and cancel it. The resulting simplified expression agrees with the original everywhere except at $x = c$ itself — which is exactly what a limit needs, since it never looks at $x = c$.

Ví dụ mẫu · Factoring and canceling

Evaluate $\lim_{x\to 3}\dfrac{x^2-9}{x^2-5x+6}$.

Direct substitution gives $\frac{0}{0}$. Factor both numerator and denominator:

$$\frac{x^2-9}{x^2-5x+6}=\frac{(x-3)(x+3)}{(x-2)(x-3)}=\frac{x+3}{x-2}\qquad(x\neq 3).$$

Now substitute $x=3$ into the simplified form: $\frac{3+3}{3-2}=\frac{6}{1}=6$. So $\lim\limits_{x\to 3}\dfrac{x^2-9}{x^2-5x+6}=6$.

(!) Lỗi thường gặp: Thấy $\frac{0}{0}$ khi thế trực tiếp **không có nghĩa là giới hạn bằng** 0, và cũng không có nghĩa là giới hạn không tồn tại. $\frac{0}{0}$ chỉ là tín hiệu cho biết biểu thức cần được **biến đổi đại số trước** (phân tích nhân tử, nhân liên hợp, quy đồng, v.v.) trước khi có thể kết luận được điều gì. Đừng bao giờ dừng lại và viết "giới hạn $=0$" ngay khi gặp dạng $\frac{0}{0}$.

1.5.2 Rationalizing using a conjugate

When a limit involves a square root and direct substitution gives $\frac{0}{0}$, multiply the numerator and denominator by the **conjugate** of the radical expression. This uses the identity $(\sqrt{A}-B)(\sqrt{A}+B)=A-B^2$ to eliminate the square root from the piece that is causing the trouble.

Ví dụ mẫu · Rationalizing with a conjugate

Evaluate $\lim\limits_{x\to 0}\dfrac{\sqrt{x+4}-2}{x}$.

Direct substitution gives $\frac{0}{0}$. Multiply by the conjugate $\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}$:

$$\frac{\sqrt{x+4}-2}{x}\cdot\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}=\frac{(x+4)-4}{x(\sqrt{x+4}+2)}=\frac{x}{x(\sqrt{x+4}+2)}=\frac{1}{\sqrt{x+4}+2}\qquad(x\neq 0).$$

Now substitute $x=0$: $\frac{1}{\sqrt{4}+2}=\frac{1}{2+2}=\frac{1}{4}$. So $\lim\limits_{x\to 0}\dfrac{\sqrt{x+4}-2}{x}=\frac{1}{4}$.

VN **GIẢI THÍCH TIẾNG VIỆT**

Khi giới hạn có chứa căn thức và cho ra dạng $\frac{0}{0}$, ta nhân cả tử và mẫu với **biểu thức liên hợp** (conjugate) của phần chứa căn — dùng hằng đẳng thức $(\sqrt{A}-B)(\sqrt{A}+B)=A-B^2$ để "phá căn". Sau khi nhân liên hợp, nhân tử gây ra 0 ở mẫu (thường là $x-c$) sẽ xuất hiện ở tử và có thể rút gọn được.

1.5.3 Combining fractions into a single expression

When a limit contains a difference or sum of fractions, combine them over a common denominator first. This often reveals a common factor that can then be canceled, exactly as in the factoring technique.

Ví dụ mẫu · Combining fractions

Evaluate $\lim\limits_{x\to 0}\dfrac{\dfrac{1}{x+2}-\dfrac{1}{2}}{x}$.

Combine the numerator over a common denominator:

$$\frac{1}{x+2}-\frac{1}{2}=\frac{2-(x+2)}{2(x+2)}=\frac{-x}{2(x+2)}.$$

Now divide by x (equivalently, multiply by $\frac{1}{x}$):

$$\frac{-x}{2(x+2)}\cdot\frac{1}{x}=\frac{-1}{2(x+2)}\qquad(x\neq 0).$$

Substitute $x=0$: $\dfrac{-1}{2(2)}=-\dfrac{1}{4}$. So $\displaystyle\lim_{x\to 0}\frac{\dfrac{1}{x+2}-\dfrac{1}{2}}{x}=-\frac{1}{4}$.

VN **GIẢI THÍCH TIẾNG VIỆT**

Khi biểu thức có chứa hiệu hoặc tổng của các phân số, hãy **quy đồng mẫu số** trước tiên rồi rút gọn tử số. Sau bước quy đồng, thường sẽ xuất hiện nhân tử chung với mẫu số bên ngoài (ở đây là x) để có thể triệt tiêu, đưa biểu thức về dạng thế trực tiếp được.

1.5.4 Limits of piecewise-defined functions

For a piecewise-defined function, the limit at a point that lies **strictly inside** one piece's domain is found by direct substitution (or algebraic simplification) using that single piece. The limit at a **breakpoint** — where the formula changes — must be found using one-sided limits, computing each side with its own piece and then comparing.

Ví dụ mẫu · A piecewise limit

Let $f(x)=\begin{cases}x^2+1, & x<2\\ 3x-1, & x\geq 2\end{cases}$. Find $\displaystyle\lim_{x\to 2}f(x)$.

Since the formula changes at $x=2$, compute both one-sided limits separately, using the piece valid on each side:

$$\lim_{x\to 2^-}f(x)=\lim_{x\to 2^-}(x^2+1)=4+1=5,\qquad \lim_{x\to 2^+}f(x)=\lim_{x\to 2^+}(3x-1)=6-1=5.$$

Both one-sided limits equal 5, so the two-sided limit exists: $\displaystyle\lim_{x\to 2}f(x)=5$. (Since $f(2)=3(2)-1=5$ also matches, f is in fact continuous at $x=2$ — a property examined closely later in this chapter.)

VN **GIẢI THÍCH TIẾNG VIỆT**

Với hàm từng khúc (piecewise), nếu điểm cần xét giới hạn nằm **hẳn bên trong** một nhánh, chỉ cần dùng đúng nhánh đó. Nhưng nếu điểm đó là **điểm nối** giữa hai nhánh (breakpoint), bắt buộc phải tính **riêng biệt** giới hạn trái (dùng nhánh bên trái) và giới hạn phải (dùng nhánh bên phải) rồi so sánh — đây là ứng dụng trực tiếp của khái niệm giới hạn một phía đã học.

1.6 Selecting Procedures for More Complex Limits

Not every limit yields to a single technique; often the right move is to recognize the *type* of expression in front of you and select — or combine — the appropriate tool.

A general strategy. (1) Try direct substitution first. (2) If it gives $\frac{0}{0}$ and the expression involves polynomials or rational functions, try factoring and canceling. (3) If it involves a square root, try rationalizing with a conjugate. (4) If it involves a sum or difference of fractions, combine them first. (5) If it involves trigonometric functions divided by x (or a multiple of x), look for the special trigonometric limits below. (6) If none of these apply directly, look for a way to bound the expression between two simpler functions with the same limit (the Squeeze Theorem, next section).

1.6.1 The special trigonometric limits

Two limits involving trigonometric functions occur so frequently that they are worth memorizing outright, since neither can be evaluated by direct substitution (both give $\frac{0}{0}$) nor by ordinary factoring:

$$\lim_{x\to 0}\frac{\sin x}{x}=1, \qquad \lim_{x\to 0}\frac{1-\cos x}{x}=0.$$

Both are believable numerically: taking $x=0.1$ radians, $\frac{\sin(0.1)}{0.1}\approx\frac{0.09983}{0.1}=0.9983$, and taking $x=0.01$, $\frac{\sin(0.01)}{0.01}\approx 0.99998$ — clearly heading toward 1. Likewise $\frac{1-\cos(0.1)}{0.1}\approx\frac{0.004996}{0.1}=0.04996$, and at $x=0.01$ this ratio is about 0.00005 — clearly heading toward 0. Graphically, both $y=\frac{\sin x}{x}$ and $y=\frac{1-\cos x}{x}$ have a hole at $x=0$ (since both are undefined there), but the curves approach the heights 1 and 0 respectively as $x\to 0$ from either side. Both facts can be proven rigorously with the Squeeze Theorem, introduced in the next section; here we treat them as established tools.

VN **GIẢI THÍCH TIẾNG VIỆT**

Hai giới hạn lượng giác đặc biệt $\lim_{x\to0}\frac{\sin x}{x}=1$ và $\lim_{x\to0}\frac{1-\cos x}{x}=0$ **không thể** tính bằng thế trực tiếp (đều cho $\frac{0}{0}$) và cũng không phân tích nhân tử được — cần ghi nhớ như công cụ có sẵn. Về mặt đồ thị, cả hai hàm đều có "lỗ hổng" tại $x=0$ nhưng đường cong vẫn tiến rõ ràng về các độ cao 1 và 0 tương ứng khi $x\to0$ từ hai phía; bảng giá trị với $x=0.1,\,0.01,\ldots$ xác nhận điều này bằng số.

1.6.2 Combining techniques

A limit involving trigonometric expressions often needs to be reshaped algebraically first — multiplying by a strategic constant, or by a conjugate-like factor — before one of the special trigonometric limits can be applied directly.

Ví dụ mẫu · Special trigonometric limits, combined with algebra

(a) Evaluate $\lim_{x\to0}\frac{\sin(4x)}{x}$. (b) Evaluate $\lim_{x\to0}\frac{1-\cos x}{x\sin x}$.

(a) Multiply and divide by 4 so the argument of sine matches the denominator:

$$\frac{\sin(4x)}{x}=4\cdot\frac{\sin(4x)}{4x}\xrightarrow[x\to0]{}4\cdot1=4.$$

(b) Multiply numerator and denominator by the conjugate-like factor $(1+\cos x)$:

$$\frac{1-\cos x}{x\sin x}\cdot\frac{1+\cos x}{1+\cos x}=\frac{1-\cos^2 x}{x\sin x(1+\cos x)}=\frac{\sin^2 x}{x\sin x(1+\cos x)}=\frac{\sin x}{x}\cdot\frac{1}{1+\cos x}.$$

As $x\to0$: $\frac{\sin x}{x}\to1$ and $\frac{1}{1+\cos x}\to\frac{1}{1+1}=\frac{1}{2}$. So the product $\to1\cdot\frac{1}{2}=\frac{1}{2}$.

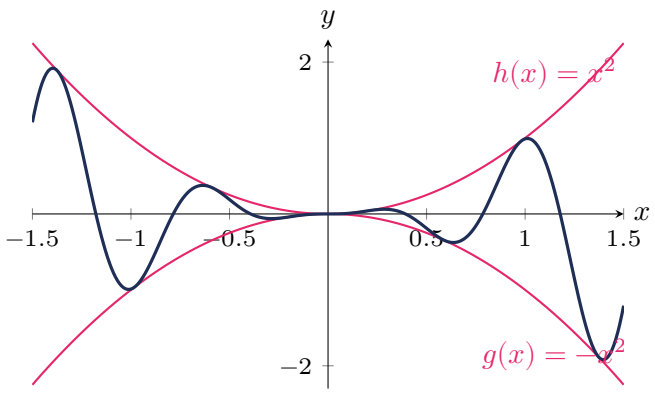
1.7 The Squeeze Theorem

Some limits cannot be evaluated directly by any algebraic simplification because the function oscillates or is otherwise too complicated to manipulate — yet it can be trapped, near the point in question, between two simpler functions that share the same limit. The **Squeeze Theorem** (also called the Sandwich Theorem) makes this idea precise.

Squeeze (Sandwich) Theorem. If $g(x) \le f(x) \le h(x)$ for all x near c (except possibly at c itself), and

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L,$$

then $\lim_{x\to c} f(x) = L$ as well — f is "squeezed" between g and h into the same limiting value.



The wiggly curve $f(x) = x^2 \sin(8x)$ is trapped between $g(x) = -x^2$ and $h(x) = x^2$ for every x, since $|\sin(8x)| \le 1$. As $x \to 0$, both bounding curves converge to 0, so $f(x)$ is squeezed to 0 as well, regardless of how wildly it oscillates elsewhere.

Ví dụ mẫu · The Squeeze Theorem in action

Evaluate $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$.

Direct substitution fails ($\sin(1/x)$ is undefined at $x = 0$ and oscillates infinitely often near 0, so this limit cannot be found by simplification). However, for every $x \neq 0$, the sine function satisfies $-1 \le \sin\left(\frac{1}{x}\right) \le 1$. Multiplying through by $x^2 \ge 0$ (which does not flip the inequalities) gives

$$-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2 \qquad \text{for all } x \neq 0.$$

Since $\lim_{x\to 0}(-x^2) = 0$ and $\lim_{x\to 0} x^2 = 0$, the Squeeze Theorem guarantees

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$

VN **GIẢI THÍCH TIẾNG VIỆT**

Định lý Kẹp dùng khi hàm số quá phức tạp (dao động, không rút gọn được) nhưng có thể "kẹp" giữa hai hàm đơn giản hơn cùng tiến về một giá trị. Ví dụ kinh điển: $\sin(1/x)$ dao động vô hạn lần khi $x \to 0$ nên không thể tính trực tiếp, nhưng vì luôn có $-1 \le \sin(1/x) \le 1$, nhân với $x^2 \ge 0$ ta kẹp được $x^2 \sin(1/x)$ giữa $-x^2$ và x^2 — cả hai đều tiến về 0, nên hàm ở giữa cũng buộc phải tiến về 0.

1.8 Connecting Graphical, Numerical, Analytical, and Verbal Representations

A single limit statement can be approached — and confirmed — through four different lenses: a **graph** (visualizing what height the curve approaches), a **table** of numerical values (watching outputs settle toward a number), an **analytical** argument (algebraic simplification or a limit law), and a **verbal** description (a sentence describing the approach in words, often arising from an applied context). Fluent use of limits means being able to translate freely among these four representations — and, when they are all available, checking that they agree.

Ví dụ mẫu · Four representations, one limit

A demographic model predicts a town's population (in thousands), t years after its founding, using $P(t) = \dfrac{t^2 - 4}{t - 2}$ for $t \neq 2$. A table of the model's predictions is shown below.

t	1.9	1.99	2.01	2.1
$P(t)$	3.9	3.99	4.01	4.1

Connect the four representations: **Numerically**, the table shows $P(t) \to 4$ as $t \to 2$ from either side. **Analytically**, factoring gives $P(t) = \dfrac{(t-2)(t+2)}{t-2} = t+2$ for $t \neq 2$, and $t+2 \to 4$ as $t \to 2$. **Graphically**, the model's graph is the line $y = t+2$ with an open circle (a hole) at $(2, 4)$, since the model provides no value at $t = 2$ itself. **Verbally**, this all says: "as the town's age approaches 2 years from either side, the model's predicted population approaches $4{,}000$ residents." All four representations agree that $\lim\limits_{t \to 2} P(t) = 4$ — even though $P(2)$ itself is undefined in the model, a removable discontinuity examined further in the next section.

VN **GIẢI THÍCH TIẾNG VIỆT**

Một giới hạn có thể được tiếp cận qua 4 hình thức: **đồ thị** (độ cao đường cong hướng tới), **bảng số** (giá trị số cụ thể tiến gần một số), **đại số/giải tích** (biến đổi công thức để thế trực tiếp), và **lời văn** (mô tả bằng ngôn ngữ tự nhiên, thường xuất phát từ bài toán thực tế). Kỹ năng quan trọng là chuyển đổi linh hoạt giữa bốn hình thức này và kiểm tra chúng có **khớp** nhau hay không — như ví dụ trên, cả bốn cách đều cho cùng một kết luận: giới hạn bằng 4.

1.9 Types of Discontinuity

A function fails to be continuous at $x = c$ in one of three characteristic ways, distinguished by what happens to the one-sided limits there.

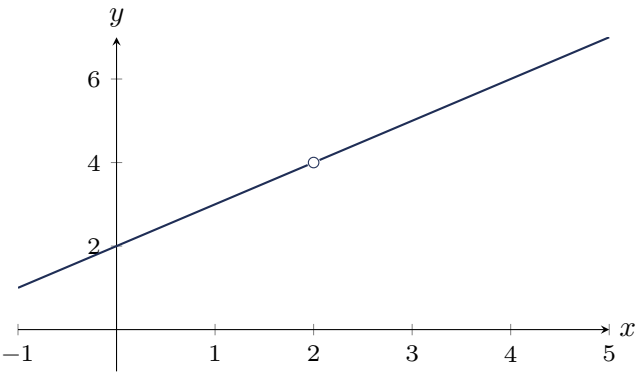
Three types of discontinuity

- **Removable discontinuity:** $\lim\limits_{x \to c} f(x)$ *exists* (both one-sided limits exist and are equal), but either $f(c)$ is undefined or $f(c)$ does not equal that limit. Graphically, a single "hole" — the discontinuity can be fixed (removed) by defining or redefining the one value $f(c)$.
- **Jump discontinuity:** both one-sided limits exist (are finite) but are *unequal*, so the two-sided limit does not exist. Graphically, the curve "jumps" from one height to another at $x = c$; no single value of $f(c)$ could ever repair this.
- **Infinite discontinuity:** at least one one-sided limit is $+\infty$ or $-\infty$. Graphically, the curve shoots off toward a vertical asymptote at $x = c$; again, no redefinition of $f(c)$ could repair this.

The table below summarizes how the three types differ in terms of one-sided limit behavior, and

— critically for the next section — whether the discontinuity can be repaired by changing a single function value.

Type	One-sided limits	Two-sided limit	Fixable by redefining $f(c)$?
Removable	Both exist and are equal	Exists (finite)	Yes
Jump	Both exist but are unequal	Does not exist	No
Infinite	At least one is $+\infty$ or $-\infty$	Does not exist	No



$g(x) = \dfrac{x^2-4}{x-2} = x+2$ for $x \neq 2$, undefined at $x = 2$. The line has a hole at $(2, 4)$: $\lim_{x\to 2} g(x) = 4$ exists, but $g(2)$ is undefined — a removable discontinuity.

Ví dụ mẫu · Classifying discontinuities

Classify the discontinuity of each function at the indicated point.

(a) $h(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ 3, & x = 1 \end{cases}$ at $x = 1$. For $x \neq 1$, $h(x) = \dfrac{(x-1)(x+1)}{x-1} = x+1$, so $\lim\limits_{x\to 1} h(x) = 2$. But $h(1) = 3 \neq 2$. The limit exists yet disagrees with the function value ⇒ **removable**.

(b) $g(x) = \begin{cases} x+1, & x < 1 \\ x^2, & x \geq 1 \end{cases}$ at $x = 1$. $\lim\limits_{x\to 1^-} g(x) = 1+1 = 2$; $\lim\limits_{x\to 1^+} g(x) = 1^2 = 1$. The one-sided limits are finite but unequal ($2 \neq 1$) ⇒ **jump**.

(c) $k(x) = \dfrac{1}{(x-3)^2}$ at $x = 3$. As $x \to 3$ from either side, $(x-3)^2 \to 0^+$, so $k(x) \to +\infty$. At least one one-sided limit is infinite ⇒ **infinite** (with a vertical asymptote at $x = 3$).

VN **GIẢI THÍCH TIẾNG VIỆT**

Ba dạng gián đoạn phân biệt bằng hành vi của giới hạn một phía: **khử được (removable)** — giới hạn tồn tại nhưng $f(c)$ sai hoặc không xác định (đồ thị có "lỗ"); **nhảy (jump)** — hai giới hạn một phía đều hữu hạn nhưng khác nhau (đồ thị "nhảy bậc"); **vô cực (infinite)** — ít nhất một giới hạn một phía là $\pm\infty$ (đồ thị có tiệm cận đứng). Chỉ loại **khử được** mới có thể "sửa" bằng cách định nghĩa lại đúng một giá trị $f(c)$.

1.10 Continuity at a Point and on an Interval

1.10.1 The formal definition of continuity

Intuitively, a function is continuous at a point if its graph can be traced through that point without lifting the pen. This intuition is captured precisely by three conditions that must *all* hold simultaneously.

f is continuous at $x = c$ if and only if all three conditions hold:

1. $f(c)$ is defined (c is in the domain of f);
2. $\lim_{x \to c} f(x)$ exists (both one-sided limits exist and are equal);
3. $\lim_{x \to c} f(x) = f(c)$ (the limit actually equals the function value).

If any one of the three fails, f is discontinuous at c.

Ví dụ mẫu · Checking the three conditions

Determine whether $f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$ is continuous at $x = 2$.

Condition 1: $f(2) = 4$ is defined. ✓

Condition 2: for $x \neq 2$, $f(x) = \dfrac{(x-2)(x+2)}{x-2} = x + 2$, so $\lim_{x \to 2} f(x) = 2 + 2 = 4$, which exists. ✓

Condition 3: the limit (4) equals $f(2)$ (4). ✓

All three conditions hold, so f **is continuous** at $x = 2$.

VN **GIẢI THÍCH TIẾNG VIỆT**

Ba điều kiện liên tục phải **đồng thời** đúng: (1) $f(c)$ xác định; (2) giới hạn tại c tồn tại; (3) giới hạn đó **bằng** $f(c)$. Chỉ cần một điều kiện sai là hàm gián đoạn tại c. Đây chính là "phiên bản chặt chẽ" của ba loại gián đoạn vừa học: removable thường vi phạm điều kiện (1) hoặc (3); jump và infinite vi phạm điều kiện (2).

1.10.2 Continuity over an interval and at endpoints

A function is continuous on an **open interval** (a, b) if it is continuous at every point inside that interval. Continuity on a **closed interval** $[a, b]$ additionally requires appropriate one-sided behavior at the two endpoints, since a two-sided limit cannot even be discussed there (there is no "outside" the interval to approach from).

Continuity on a closed interval

f is continuous on $[a, b]$ if: f is continuous at every point of the open interval (a, b); **and** f is *right-continuous* at a, meaning $\lim_{x \to a^+} f(x) = f(a)$; **and** f is *left-continuous* at b, meaning $\lim_{x \to b^-} f(x) = f(b)$.

Ví dụ mẫu · Continuity on a closed interval

Explain why $f(x) = \sqrt{4 - x^2}$ is continuous on the closed interval $[-2, 2]$.

On the open interval $(-2, 2)$, f is a composition of the continuous function $4 - x^2$ (positive there) with the continuous square-root function, so f is continuous at every interior point by direct substitution. At the left endpoint $x = -2$, only the right-hand limit makes sense (there is no domain to the left of -2): $\lim_{x \to -2^+} \sqrt{4 - x^2} = \sqrt{4 - 4} = 0 = f(-2)$, so f is right-continuous at -2. At the right endpoint $x = 2$, only the left-hand limit makes sense: $\lim_{x \to 2^-} \sqrt{4 - x^2} = \sqrt{4 - 4} = 0 = f(2)$, so f is left-continuous at 2. All requirements are satisfied, so f is continuous on $[-2, 2]$.

VN **GIẢI THÍCH TIẾNG VIỆT**
Liên tục trên khoảng **mở** (a, b): chỉ cần liên tục tại mọi điểm bên trong. Liên tục trên đoạn **đóng** $[a, b]$: cần thêm liên tục **một phía** tại hai đầu mút — liên tục phải tại a (vì không có gì bên trái a để so sánh) và liên tục trái tại b (vì không có gì bên phải b). Đây là điểm học sinh hay bỏ sót khi xét các hàm có miền xác định giới hạn, ví dụ hàm căn.

(!) Lỗi thường gặp: Khi kiểm tra tính liên tục tại một điểm nối (breakpoint) của hàm từng khúc, một lỗi rất phổ biến là chỉ tính giới hạn từ **một phía** rồi kết luận vội. Luôn phải kiểm tra **cả hai** giới hạn một phía, xác nhận chúng bằng nhau, *rồi mới* so sánh với $f(c)$. Bỏ sót một phía có thể khiến ta kết luận sai một hàm không liên tục là liên tục (nếu tình cờ chỉ phía đã kiểm tra khớp với $f(c)$), hoặc ngược lại.

1.11 Removing Discontinuities

When a function has a **removable** discontinuity at $x = c$ — meaning $\lim_{x \to c} f(x)$ exists even though $f(c)$ is undefined or wrong — a new function can be built that agrees with f everywhere except at c, and is assigned the value of the limit exactly at c. This new function is continuous at c; informally, we say the discontinuity has been "removed." This is precisely why this type of discontinuity earns the name *removable* — unlike jump or infinite discontinuities, which no single redefinition of one point could ever fix.

Ví dụ mẫu · Removing a discontinuity

The function $f(x) = \dfrac{x^2 - 5x + 6}{x - 3}$ is undefined at $x = 3$. Find the value k such that defining $f(3) = k$ produces a function that is continuous at $x = 3$.

Factor the numerator: $x^2 - 5x + 6 = (x - 2)(x - 3)$, so for $x \neq 3$,

$$f(x) = \frac{(x - 2)(x - 3)}{x - 3} = x - 2.$$

Then $\lim_{x \to 3} f(x) = 3 - 2 = 1$. For the redefined function to be continuous at $x = 3$, condition (3) of the continuity definition requires $f(3)$ to equal this limit, so $k = 1$.

VN **GIẢI THÍCH TIẾNG VIỆT**
"Khử" một gián đoạn nghĩa là **định nghĩa lại đúng một giá trị** $f(c)$ sao cho nó bằng giới hạn $\lim_{x \to c} f(x)$ đã tính được — khi đó hàm mới thoả cả ba điều kiện liên tục tại c. Kỹ thuật luôn giống nhau: rút gọn biểu thức (thường bằng phân tích nhân tử) để tìm giới hạn, rồi gán giá trị đó cho $f(c)$.

1.12 Infinite Limits and Vertical Asymptotes

When $f(x)$ grows without bound as x approaches c from one or both sides, we write $\lim_{x \to c} f(x) = +\infty$ (or $-\infty$). Strictly speaking, this limit still "does not exist" as a finite number — but the $\pm\infty$ notation is far more informative than simply saying DNE, since it describes precisely *how* the function fails to settle down.

Vertical asymptotes

The line $x = c$ is a **vertical asymptote** of f if at least one of $\lim_{x\to c^-} f(x)$ or $\lim_{x\to c^+} f(x)$ is $+\infty$ or $-\infty$. For a rational function $\dfrac{p(x)}{q(x)}$ in lowest terms, vertical asymptotes occur exactly at the real zeros of $q(x)$ (points where the denominator vanishes but the numerator does not).

If $p(x)$ and $q(x)$ share a common factor that vanishes at $x = c$ (so the fraction is *not* in lowest terms there), that zero of the denominator produces a removable discontinuity instead of a vertical asymptote — exactly the situation analyzed algebraically in the factoring technique earlier in this chapter. Always cancel common factors first; only the zeros of the denominator that *survive* cancellation give genuine vertical asymptotes.

To determine the *sign* of an infinite one-sided limit, examine the sign of the numerator and the sign of the denominator separately as x approaches c from that side, then apply the usual division sign rules.

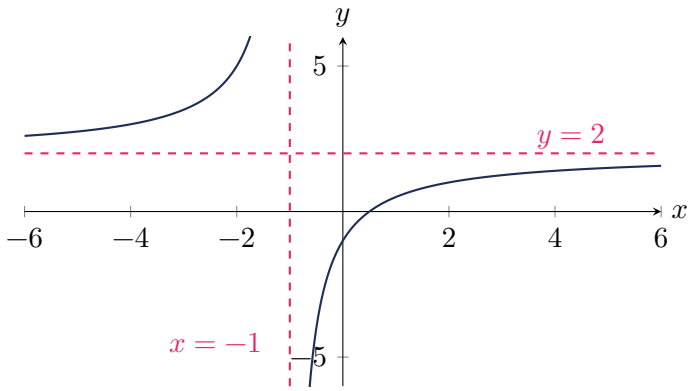
Ví dụ mẫu · Vertical asymptote sign analysis

Analyze $\lim_{x\to 4^-} \dfrac{3}{x-4}$ and $\lim_{x\to 4^+} \dfrac{3}{x-4}$.

The numerator is the constant $3 > 0$ throughout. As $x \to 4^-$ (e.g., $x = 3.99$), the denominator $x - 4 \to 0^-$ (a small negative number), so $\dfrac{3}{x-4} \to \dfrac{3}{0^-} = -\infty$. As $x \to 4^+$ (e.g., $x = 4.01$), the denominator $x - 4 \to 0^+$ (a small positive number), so $\dfrac{3}{x-4} \to \dfrac{3}{0^+} = +\infty$. Thus

$$\lim_{x\to 4^-} \frac{3}{x-4} = -\infty, \qquad \lim_{x\to 4^+} \frac{3}{x-4} = +\infty,$$

and $x = 4$ is a vertical asymptote across which the function's sign flips from negative to positive.



$f(x) = \dfrac{2x-1}{x+1}$: a vertical asymptote at $x = -1$ (denominator zero, numerator $-3 \neq 0$ there), with $f(x) \to +\infty$ as $x \to -1^-$ and $f(x) \to -\infty$ as $x \to -1^+$; and a horizontal asymptote $y = 2$, approached as $x \to \pm\infty$ (discussed in the next section).

(!) Lỗi thường gặp: Xác định **dấu** của giới hạn vô cực một phía là bước dễ sai nhất. Cách an toàn nhất: chọn một giá trị x cụ thể, rất gần c về đúng phía đang xét (ví dụ $x = 3.99$ cho $x \to 4^-$), thế vào và xem **dấu** của tử và mẫu riêng biệt — đừng suy luận "chay" trong đầu. Một lỗi thường gặp là mặc định cả hai giới hạn một phía tại cùng một tiệm cận đứng phải **cùng dấu**; trong ví dụ trên, dấu đổi từ $-\infty$ sang $+\infty$ ngay tại $x = 4$.

1.13 Limits at Infinity and Horizontal Asymptotes

A different question asks what happens to $f(x)$ as x itself grows without bound, in either the positive or negative direction. We write $\lim_{x\to\infty} f(x) = L$ (or $\lim_{x\to-\infty} f(x) = L$) to mean $f(x)$ gets arbitrarily close to the finite number L as x increases (or decreases) without bound. When such a limit exists, the line $y = L$ is a **horizontal asymptote** of f, describing the function's long-run **end behavior**.

Two different uses of ∞

It is easy to conflate two independent ideas that both involve the symbol ∞. An **infinite limit** (previous section) asks what happens to the *output* $f(x)$ as the input x approaches a specific, finite number c — this is behavior in the vertical direction, near a vertical asymptote. A **limit at infinity** asks what happens to the output $f(x)$ as the *input* x itself grows without bound — this is behavior in the horizontal direction, far to the left or right, near a horizontal asymptote. A single function can have vertical asymptotes, horizontal asymptotes, both, or neither; the two questions are answered by entirely different reasoning, and neither implies the other.

1.13.1 End behavior of rational functions

For a rational function $\dfrac{p(x)}{q(x)}$, end behavior is governed entirely by comparing the degree n of the numerator $p(x)$ to the degree m of the denominator $q(x)$; dividing numerator and denominator by the highest power of x present makes the pattern clear.

End behavior of rational functions. Let $n = \deg p$, $m = \deg q$.

- If $n < m$: $\lim_{x\to\pm\infty} \dfrac{p(x)}{q(x)} = 0$; horizontal asymptote $y = 0$.
- If $n = m$: $\lim_{x\to\pm\infty} \dfrac{p(x)}{q(x)} = \dfrac{a_n}{b_m}$, the ratio of the leading coefficients; horizontal asymptote $y = \dfrac{a_n}{b_m}$.
- If $n > m$: the limit is $+\infty$ or $-\infty$ (no finite value); **there is no horizontal asymptote.**

Ví dụ mẫu · End behavior of rational functions

Evaluate each limit and identify the horizontal asymptote, if any. (a) $\lim_{x\to\infty} \dfrac{2x+1}{x^2-4}$. (b) $\lim_{x\to\infty} \dfrac{3x^2-5x+1}{6x^2+x-7}$. (c) $\lim_{x\to\infty} \dfrac{x^3-1}{2x^2+5}$.

(a) Numerator degree $1 <$ denominator degree 2. Dividing every term by x^2: $\dfrac{2/x + 1/x^2}{1 - 4/x^2} \to \dfrac{0+0}{1-0} = 0$. Horizontal asymptote $y = 0$.

(b) Both degrees equal 2. Dividing every term by x^2: $\dfrac{3 - 5/x + 1/x^2}{6 + 1/x - 7/x^2} \to \dfrac{3-0+0}{6+0-0} = \dfrac{3}{6} = \dfrac{1}{2}$. Horizontal asymptote $y = \dfrac{1}{2}$.

(c) Numerator degree $3 >$ denominator degree 2. The numerator dominates: as $x \to \infty$, $\dfrac{x^3-1}{2x^2+5} \approx \dfrac{x^3}{2x^2} = \dfrac{x}{2} \to \infty$. The limit is $+\infty$; there is no horizontal asymptote.

VN **GIẢI THÍCH TIẾNG VIỆT**

Với hàm phân thức, so sánh bậc tử (n) và bậc mẫu (m): $n < m \Rightarrow$ tiệm cận ngang $y = 0$; $n = m \Rightarrow y = \dfrac{a_n}{b_m}$ (tỉ số hệ số bậc cao nhất); $n > m \Rightarrow$ không có tiệm cận ngang (hàm tăng/giảm không giới hạn). Mẹo chứng minh: chia cả tử và mẫu cho luỹ thừa bậc cao nhất của x ở mẫu, mọi số hạng dạng $\frac{1}{x^k}$ ($k > 0$) sẽ tiến về 0 khi $x \to \pm\infty$.

1.13.2 End behavior of functions built from exponentials

Exponential functions dominate polynomials at infinity in a way that produces distinctive end behavior. The key facts are $\lim\limits_{x\to\infty} e^x = +\infty$ and $\lim\limits_{x\to-\infty} e^x = 0$. For a ratio built from exponentials, divide through by the fastest-growing exponential term present, exactly as with rational functions.

Ví dụ mẫu · End behavior with exponentials

Evaluate $\lim\limits_{x\to\infty} \dfrac{5e^x - 3}{2e^x + 7}$ and $\lim\limits_{x\to-\infty} \dfrac{5e^x - 3}{2e^x + 7}$.

As $x \to \infty$: divide numerator and denominator by e^x (the dominant term):

$$\frac{5e^x - 3}{2e^x + 7} = \frac{5 - 3e^{-x}}{2 + 7e^{-x}}.$$

As $x \to \infty$, $e^{-x} \to 0$, so this ratio $\to \dfrac{5 - 0}{2 + 0} = \dfrac{5}{2}$.

As $x \to -\infty$: here $e^x \to 0$ directly, so substitute $e^x = 0$ into the original expression:

$$\frac{5(0) - 3}{2(0) + 7} = \frac{-3}{7}.$$

So $\lim\limits_{x\to\infty} \dfrac{5e^x - 3}{2e^x + 7} = \dfrac{5}{2}$ while $\lim\limits_{x\to-\infty} \dfrac{5e^x - 3}{2e^x + 7} = -\dfrac{3}{7}$ — **two different** horizontal asymptotes, one for each direction.

VN **GIẢI THÍCH TIẾNG VIỆT**

Với hàm dựng từ e^x, dùng $\lim_{x\to\infty} e^x = +\infty$ và $\lim_{x\to-\infty} e^x = 0$. Kỹ thuật giống hàm phân thức: chia cho số hạng e^x chiếm ưu thế nhất (khi $x \to \infty$) hoặc thay trực tiếp $e^x \to 0$ (khi $x \to -\infty$). Điểm khác biệt quan trọng so với hàm phân thức: một hàm dựng từ e^x hoàn toàn có thể có **hai tiệm cận ngang khác nhau** — một khi $x \to +\infty$ và một khi $x \to -\infty$ — trong khi hàm phân thức (khi có tiệm cận ngang) luôn cho cùng một giá trị ở cả hai phía.

(!) Lỗi thường gặp: Đừng nhầm lẫn tiệm cận ngang với tiệm cận đứng: đồ thị **không bao giờ** chạm hoặc vượt qua tiệm cận **đứng** (vì hàm không xác định tại đó), nhưng đồ thị **hoàn toàn có thể** chạm hoặc cắt ngang qua tiệm cận **ngang** tại một vài điểm hữu hạn — tiệm cận ngang chỉ mô tả hành vi khi $x \to \pm\infty$, không phải một "ranh giới cấm" trên toàn miền xác định. Ví dụ $f(x) = \dfrac{\sin x}{x}$ có tiệm cận ngang $y = 0$ nhưng cắt trục hoành ($y = 0$) vô số lần.

The table below collects the end-behavior rules from this section into a single quick-reference summary.

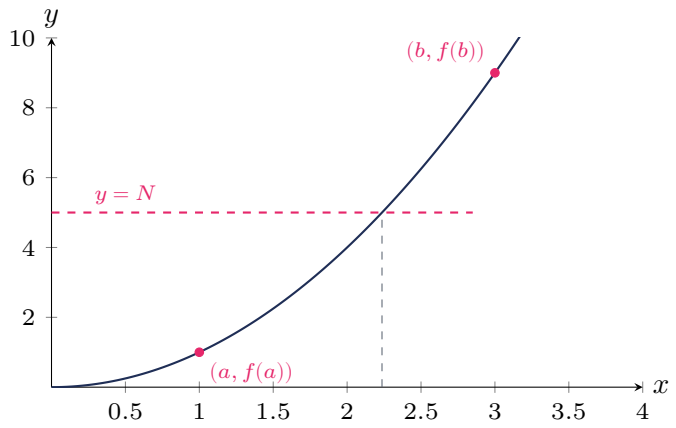
Function type	As $x \to +\infty$	As $x \to -\infty$
Rational, $n < m$	$\to 0$	$\to 0$ (same on both sides)
Rational, $n = m$	$\to a_n/b_m$	$\to a_n/b_m$ (same on both sides)
Rational, $n > m$	$\to +\infty$ or $-\infty$ (no HA)	$\to +\infty$ or $-\infty$ (no HA)
Built from e^x	determined by the dominant e^x term	substitute $e^x \to 0$ directly

1.14 The Intermediate Value Theorem

Continuity has a powerful geometric consequence: the graph of a continuous function cannot "skip over" any y-value between two of its outputs — to get from one height to another, a continuous curve must pass through every height in between.

Intermediate Value Theorem (IVT). If f is continuous on the closed interval $[a, b]$, and N is any number between $f(a)$ and $f(b)$, then there exists at least one $c \in (a, b)$ such that $f(c) = N$.

Geometrically, if you draw a horizontal line $y = N$ at any height strictly between $f(a)$ and $f(b)$, that line must cross the graph of f at least once somewhere between $x = a$ and $x = b$ — continuity leaves no way to jump over it. It is essential to understand what the IVT does *not* say: it **guarantees the existence** of at least one such c, but it does not say how many such points exist (there could be several), and it gives no method for actually **locating** c exactly.



$f(x) = x^2$ on $[a, b] = [1, 3]$: $f(1) = 1$ and $f(3) = 9$. Since $N = 5$ lies between 1 and 9, IVT guarantees some $c \in (1, 3)$ with $f(c) = 5$; indeed $c = \sqrt{5} \approx 2.236$.

Ví dụ mẫu · Using the Intermediate Value Theorem

(a) Show that $f(x) = x^3 - x - 1$ has at least one real root in the interval $(1, 2)$.

f is a polynomial, so it is continuous everywhere, in particular on $[1, 2]$. Evaluate the endpoints:

$$f(1) = 1 - 1 - 1 = -1, \qquad f(2) = 8 - 2 - 1 = 5.$$

Since $f(1) = -1 < 0 < 5 = f(2)$, the value $N = 0$ lies between $f(1)$ and $f(2)$. By the IVT, there exists $c \in (1, 2)$ with $f(c) = 0$ — that is, a root of f in $(1, 2)$. (The IVT confirms a root exists but does not tell us its exact value.)

(b) A tank of water is heated over 10 minutes; its temperature $T(t)$ (in °C) is a continuous function of time $t \in [0, 10]$, with $T(0) = 20$ and $T(10) = 310$. Explain why the water must, at some moment, have had a temperature of exactly 150°C.

T is continuous on $[0, 10]$ (given), and $N = 150$ satisfies $T(0) = 20 < 150 < 310 = T(10)$. By the IVT, there exists some $t \in (0, 10)$ with $T(t) = 150$ — the water's temperature must have passed through 150°C at some instant, even though we cannot say exactly when from

this information alone.

VN **GIẢI THÍCH TIẾNG VIỆT**

Định lý Giá trị Trung gian (IVT) nói rằng: nếu f liên tục trên $[a, b]$, thì với **mọi** giá trị N nằm giữa $f(a)$ và $f(b)$, tồn tại ít nhất một điểm $c \in (a, b)$ sao cho $f(c) = N$. Về hình học: đường cong liên tục không thể "nhảy qua" một độ cao nào đó mà không cắt ngang nó. Điều quan trọng cần nhớ: IVT chỉ **đảm bảo sự tồn tại** của c — không cho biết có bao nhiêu điểm c như vậy, và không cho cách **tìm** ra giá trị c chính xác.

1.15 Problem Bank

Bài tập luyện tập

A ball's height is $h(t) = -16t^2 + 64t + 5$ feet, t in seconds. Find the average velocity of the ball on the interval $[1, 3]$.

Lời giải chi tiết

$h(1) = -16 + 64 + 5 = 53$ and $h(3) = -16(9) + 64(3) + 5 = -144 + 192 + 5 = 53$. Average velocity $= \dfrac{h(3) - h(1)}{3 - 1} = \dfrac{53 - 53}{2} = \dfrac{0}{2} = 0$ ft/s. The ball's heights at $t = 1$ and $t = 3$ happen to coincide (the parabola is symmetric about its vertex at $t = 2$), so despite rising and falling in between, its net average rate of change over $[1, 3]$ is exactly 0.

Bài tập luyện tập

For $s(t) = t^3 - 3t$, the instantaneous velocity at $t = 2$, found as $\lim\limits_{h \to 0} \dfrac{s(2 + h) - s(2)}{h}$, equals:

(A) 2 **(C)** 9

(B) 6 **(D)** 12

Lời giải chi tiết

$s(2) = 8 - 6 = 2$. Expand $s(2 + h) = (2 + h)^3 - 3(2 + h) = 8 + 12h + 6h^2 + h^3 - 6 - 3h = 2 + 9h + 6h^2 + h^3$. So

$$\frac{s(2 + h) - s(2)}{h} = \frac{9h + 6h^2 + h^3}{h} = 9 + 6h + h^2 \xrightarrow[h \to 0]{} 9.$$

(C).

Bài tập luyện tập

Let $f(x) = \begin{cases} 2x + 1, & x < 0 \\ x^2 - 1, & x \geq 0 \end{cases}$. Find $\lim\limits_{x \to 0^-} f(x)$ and $\lim\limits_{x \to 0^+} f(x)$, and state whether $\lim\limits_{x \to 0} f(x)$ exists.

Lời giải chi tiết

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(2x+1) = 2(0)+1 = 1$. $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x^2-1) = 0-1 = -1$. Since $1 \neq -1$, the one-sided limits disagree, so $\lim_{x\to 0} f(x)$ **does not exist**.

Bài tập luyện tập

Suppose $\lim_{x\to 3^-} f(x) = 5$ and $\lim_{x\to 3^+} f(x) = 5$, but $f(3) = 2$. Which statement is true?

(A) $\lim_{x\to 3} f(x)$ does not exist

(B) $\lim_{x\to 3} f(x) = 2$

(C) $\lim_{x\to 3} f(x) = 5$

(D) f is continuous at $x = 3$

Lời giải chi tiết

Since both one-sided limits exist and agree ($5 = 5$), the two-sided limit exists and equals 5, regardless of $f(3)$. **(C)**. Note that f is *not* continuous at $x = 3$, since $f(3) = 2 \neq 5 = \lim_{x\to 3} f(x)$ — this is a removable discontinuity, so (D) is false.

Bài tập luyện tập

Estimate $\lim_{x\to 3} \dfrac{x^3-27}{x-3}$ using a table of values at $x = 2.9,\ 2.99,\ 2.999,\ 3.001,\ 3.01$, then confirm your estimate algebraically.

Lời giải chi tiết

Algebraically, $x^3-27 = (x-3)(x^2+3x+9)$, so for $x \neq 3$, $\dfrac{x^3-27}{x-3} = x^2+3x+9$. The table values are: $f(2.9) = 8.41+8.7+9 = 26.11$; $f(2.99) \approx 26.9101$; $f(2.999) \approx 26.9910$; $f(3.001) \approx 27.0090$; $f(3.01) \approx 27.0901$ — clearly approaching 27. Algebraically, $x^2+3x+9 \to 9+9+9 = 27$ as $x \to 3$, confirming $\lim_{x\to 3} \dfrac{x^3-27}{x-3} = 27$.

Bài tập luyện tập

Suppose $\lim_{x\to 5} p(x) = 4$ and $\lim_{x\to 5} q(x) = 13$. Find $\lim_{x\to 5}\left[p(x)q(x)+5\right]$ and $\lim_{x\to 5}\sqrt{q(x)-p(x)}$.

Lời giải chi tiết

By the product and sum laws: $\lim\left[p(x)q(x)+5\right] = (4)(13)+5 = 52+5 = 57$. By the difference and power laws: $\lim\sqrt{q(x)-p(x)} = \sqrt{13-4} = \sqrt{9} = 3$.

Bài tập luyện tập

Evaluate $\lim_{x\to 2} \dfrac{x^3+2x-1}{x^2+1}$ by direct substitution.

(A) $\dfrac{11}{5}$

(B) $\dfrac{5}{11}$

(C) 2

(D) 11

Lời giải chi tiết

The function is a quotient of polynomials, continuous at $x = 2$ since $2^2 + 1 = 5 \neq 0$. Substitute directly: $\frac{8+4-1}{4+1} = \frac{11}{5}$. **(A)**.

Bài tập luyện tập

Evaluate $\lim_{x \to -2} \frac{x^2 + 5x + 6}{x^2 - 4}$.

Lời giải chi tiết

Factor: $\frac{(x+2)(x+3)}{(x-2)(x+2)} = \frac{x+3}{x-2}$ for $x \neq -2$. Substitute $x = -2$: $\frac{-2+3}{-2-2} = \frac{1}{-4} = -\frac{1}{4}$.

Bài tập luyện tập

Evaluate $\lim_{x \to 4} \frac{x^2 - 16}{x^2 - 3x - 4}$.

(A) $\frac{8}{5}$

(B) $\frac{5}{8}$

(C) 0

(D) DNE

Lời giải chi tiết

Factor: $\frac{(x-4)(x+4)}{(x-4)(x+1)} = \frac{x+4}{x+1}$ for $x \neq 4$. Substitute $x = 4$: $\frac{8}{5}$. **(A)**.

Bài tập luyện tập

Evaluate $\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}$.

Lời giải chi tiết

Multiply by the conjugate $\frac{\sqrt{x}+3}{\sqrt{x}+3}$: $\frac{x-9}{(x-9)(\sqrt{x}+3)} = \frac{1}{\sqrt{x}+3}$ for $x \neq 9$. Substitute $x = 9$: $\frac{1}{3+3} = \frac{1}{6}$.

Bài tập luyện tập

Evaluate $\lim_{x \to 0} \frac{\frac{1}{3+x} - \frac{1}{3}}{x}$.

Lời giải chi tiết

Combine: $\frac{1}{3+x} - \frac{1}{3} = \frac{3-(3+x)}{3(3+x)} = \frac{-x}{3(3+x)}$. Divide by x: $\frac{-1}{3(3+x)}$. Substitute $x = 0$: $\frac{-1}{9}$.

Bài tập luyện tập

Let $f(x) = \begin{cases} 3x-2, & x \le 1 \\ x^2, & x > 1 \end{cases}$. Then $\lim_{x\to 1} f(x) =$

(A) 1
(C) -2
(B) DNE
(D) 0

Lời giải chi tiết

$\lim_{x\to 1^-} f(x) = 3(1) - 2 = 1$. $\lim_{x\to 1^+} f(x) = 1^2 = 1$. Both equal 1, so the limit is 1. **(A)**.

Bài tập luyện tập

Evaluate $\lim_{x\to 0} \dfrac{\sin(7x)}{\sin(2x)}$.

Lời giải chi tiết

Rewrite as $\dfrac{\sin(7x)}{7x} \cdot \dfrac{7x}{2x} \cdot \dfrac{2x}{\sin(2x)}$. As $x \to 0$, the first and last factors each $\to 1$, leaving $\dfrac{7x}{2x} = \dfrac{7}{2}$. So the limit is $\dfrac{7}{2}$.

Bài tập luyện tập

Evaluate $\lim_{x\to 0} \dfrac{1-\cos(3x)}{x}$.

(A) 0
(C) $\dfrac{1}{3}$
(B) 3
(D) DNE

Lời giải chi tiết

Write $\dfrac{1-\cos(3x)}{x} = 3 \cdot \dfrac{1-\cos(3x)}{3x}$. As $x \to 0$, letting $u = 3x \to 0$, $\dfrac{1-\cos u}{u} \to 0$, so the whole expression $\to 3 \cdot 0 = 0$. **(A)**.

Bài tập luyện tập

Suppose $5 - x^2 \le f(x) \le 5 + x^2$ for all x near 0. Find $\lim_{x\to 0} f(x)$, and name the theorem used.

Lời giải chi tiết

Both bounding functions satisfy $\lim_{x\to 0}(5 - x^2) = 5$ and $\lim_{x\to 0}(5 + x^2) = 5$. Since $f(x)$ is trapped between them near 0, the Squeeze (Sandwich) Theorem guarantees $\lim_{x\to 0} f(x) = 5$.

Bài tập luyện tập

A table of values for $r(x) = x^2 + 5$ near $x = 2$ gives $r(1.9) = 8.61$, $r(1.99) = 8.9601$, $r(2.01) = 9.0401$, $r(2.1) = 9.41$. (a) Estimate $\lim_{x\to 2} r(x)$ from the table. (b) Confirm analytically. (c) Is r continuous at $x = 2$? Justify using the three-condition definition.

Lời giải chi tiết

(a) The table values approach 9 from both sides, so the estimate is $\lim_{x\to 2} r(x) \approx 9$.
(b) r is a polynomial, continuous everywhere, so $\lim_{x\to 2} r(x) = r(2) = 4 + 5 = 9$ by direct substitution — matching the table.
(c) Yes. $r(2) = 9$ is defined (condition 1); $\lim_{x\to 2} r(x) = 9$ exists (condition 2); and this limit equals $r(2) = 9$ (condition 3). All three conditions hold, so r is continuous at $x = 2$.

Bài tập luyện tập

Classify the discontinuity of $w(x) = \dfrac{x+5}{x^2-25}$ at (a) $x = 5$ and (b) $x = -5$.

Lời giải chi tiết

Factor the denominator: $x^2 - 25 = (x-5)(x+5)$.
(a) At $x = 5$: the numerator $x + 5 = 10 \neq 0$ while the denominator $\to 0$, so $w(x) \to \pm\infty$ — an **infinite discontinuity** (vertical asymptote) at $x = 5$.
(b) At $x = -5$: both numerator and denominator vanish, and the common factor cancels: $w(x) = \dfrac{x+5}{(x-5)(x+5)} = \dfrac{1}{x-5}$ for $x \neq -5$, giving a finite limit $\lim_{x\to -5} w(x) = \dfrac{1}{-10} = -\dfrac{1}{10}$, while $w(-5)$ itself is undefined — a **removable discontinuity** at $x = -5$.

Bài tập luyện tập

Let $m(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$. At $x = 0$, m has:

(A) A removable discontinuity
(B) A jump discontinuity
(C) An infinite discontinuity
(D) No discontinuity — m is continuous at 0

Lời giải chi tiết

For $x < 0$, $\dfrac{|x|}{x} = \dfrac{-x}{x} = -1$; for $x > 0$, $\dfrac{|x|}{x} = \dfrac{x}{x} = 1$. So $\lim_{x\to 0^-} m(x) = -1$ and $\lim_{x\to 0^+} m(x) = 1$: both finite but unequal. **(B).**

Bài tập luyện tập

A function q satisfies $\lim_{x\to -1^-} q(x) = 3$, $\lim_{x\to -1^+} q(x) = 3$, and $q(-1) = -2$. What is true at $x = -1$?

(A) q is continuous there
(B) q has a jump discontinuity
(C) q has a removable discontinuity
(D) $\lim_{x\to -1} q(x)$ does not exist

Lời giải chi tiết

The one-sided limits agree ($3 = 3$), so the two-sided limit exists: $\lim_{x\to -1} q(x) = 3$ — this rules out (B) and (D). But $q(-1) = -2 \neq 3$, so condition (3) of continuity fails and q is *not* continuous — this rules out (A). Since the limit exists but disagrees with the function value, this is exactly a **removable discontinuity**. **(C)**.

Bài tập luyện tập

Explain why $f(x) = \sqrt{4 - x^2}$ is continuous on the closed interval $[-2, 2]$, addressing both endpoints.

Lời giải chi tiết

On $(-2, 2)$, f is a composition of the continuous functions $4 - x^2$ and $\sqrt{\ }$, hence continuous at every interior point by direct substitution. At $x = -2$, only $\lim_{x\to -2^+} f(x) = \sqrt{4-4} = 0 = f(-2)$ needs to hold (right-continuity), and it does. At $x = 2$, only $\lim_{x\to 2^-} f(x) = \sqrt{4-4} = 0 = f(2)$ needs to hold (left-continuity), and it does. Since f is continuous on the interior and one-sided continuous at each endpoint, f is continuous on $[-2, 2]$.

Bài tập luyện tập

Find the value of c that makes $g(x) = \begin{cases} \dfrac{x^2 + x - 6}{x - 2}, & x \neq 2 \\ c, & x = 2 \end{cases}$ continuous at $x = 2$.

Lời giải chi tiết

Factor: $x^2 + x - 6 = (x + 3)(x - 2)$, so $g(x) = x + 3$ for $x \neq 2$. Then $\lim_{x\to 2} g(x) = 2 + 3 = 5$. For continuity, c must equal this limit: $c = 5$.

Bài tập luyện tập

For $p(x) = \dfrac{-2}{x - 1}$, the pair $(\lim_{x\to 1^-} p(x),\ \lim_{x\to 1^+} p(x))$ is:

(A) $(+\infty, +\infty)$ **(C)** $(+\infty, -\infty)$

(B) $(-\infty, -\infty)$ **(D)** $(-\infty, +\infty)$

Lời giải chi tiết

As $x \to 1^-$ (e.g., $x = 0.99$), $x - 1 \to 0^-$, so $\dfrac{-2}{x-1} \to \dfrac{-2}{0^-} = +\infty$. As $x \to 1^+$ (e.g., $x = 1.01$), $x - 1 \to 0^+$, so $\dfrac{-2}{x-1} \to \dfrac{-2}{0^+} = -\infty$. **(C)**.

Bài tập luyện tập

Evaluate $\lim_{x\to\infty} \dfrac{7x^2 - 1}{2x^2 + 3x}$.

(A) $\frac{7}{2}$
(B) 0
(C) ∞
(D) $\frac{7}{3}$

Lời giải chi tiết

Degrees of numerator and denominator are both 2. Divide by x^2: $\frac{7-1/x^2}{2+3/x} \to \frac{7}{2}$. **(A)**.

Bài tập luyện tập

Evaluate $\lim_{x\to\infty} \frac{4x^3-2x}{x^2+1}$ and state whether a horizontal asymptote exists.

Lời giải chi tiết

Numerator degree $3 >$ denominator degree 2, so no finite limit exists. Since $\frac{4x^3-2x}{x^2+1} \approx \frac{4x^3}{x^2} = 4x \to \infty$ as $x \to \infty$, the limit is $+\infty$, and there is **no horizontal asymptote**.

Bài tập luyện tập

Find all vertical and horizontal asymptotes of $f(x) = \frac{2x^2-8}{x^2-x-6}$, and identify any removable discontinuity (hole).

Lời giải chi tiết

Factor: numerator $2x^2-8 = 2(x-2)(x+2)$; denominator $x^2-x-6 = (x-3)(x+2)$. The common factor $(x+2)$ cancels:

$$f(x) = \frac{2(x-2)(x+2)}{(x-3)(x+2)} = \frac{2(x-2)}{x-3} \qquad (x \neq -2).$$

At $x=-2$: this factor canceled, so there is a **removable discontinuity** (a hole), not a vertical asymptote, at $\left(-2,\ \frac{2(-4)}{-5}\right) = \left(-2,\ \frac{8}{5}\right)$. At $x=3$: the denominator vanishes but the numerator does not ($2(3-2)=2\neq 0$), so $x=3$ is a genuine **vertical asymptote**. For the horizontal asymptote, both the original numerator and denominator have degree 2, so the ratio of leading coefficients gives $y=\frac{2}{1}=2$ (confirmed by dividing the simplified form: $\frac{2(x-2)}{x-3} \to \frac{2x}{x} = 2$ as $x \to \pm\infty$). **Summary:** vertical asymptote $x=3$; horizontal asymptote $y=2$; hole at $\left(-2, \frac{8}{5}\right)$.

Bài tập luyện tập

Evaluate $\lim_{x\to-\infty} \frac{3+e^x}{1-e^x}$.

(A) 3
(B) -3
(C) 1
(D) 0

Lời giải chi tiết

As $x \to -\infty$, $e^x \to 0$. Substituting: $\dfrac{3+0}{1-0} = 3$. **(A)**.

Bài tập luyện tập

Show that $f(x) = x^5 - 3x^2 + 1$ has at least one real root in $(0, 1)$.

Lời giải chi tiết

f is a polynomial, continuous on $[0, 1]$. $f(0) = 0 - 0 + 1 = 1 > 0$. $f(1) = 1 - 3 + 1 = -1 < 0$. Since $f(0) = 1$ and $f(1) = -1$ have opposite signs, $N = 0$ lies between them, so by the IVT there exists $c \in (0, 1)$ with $f(c) = 0$ — a root of f in $(0, 1)$.

Bài tập luyện tập

A rock's temperature $T(t)$ (in °C) is continuous on $[0, 10]$ minutes, with $T(0) = 20$ and $T(10) = 310$. Which of the following *must* be true by the Intermediate Value Theorem?

(A) $T(t) = 20$ for some $t \in (0, 10)$

(B) $T(t) = 150$ for some $t \in (0, 10)$

(C) T is increasing on $[0, 10]$

(D) $T(5) = 165$

Lời giải chi tiết

Since 150 lies strictly between $T(0) = 20$ and $T(10) = 310$, and T is continuous on $[0, 10]$, the IVT guarantees some $t \in (0, 10)$ with $T(t) = 150$. **(B)**. Choice (A) is not guaranteed (the temperature need not return to 20); (C) is not guaranteed (IVT says nothing about monotonicity); (D) is not guaranteed (IVT does not pinpoint where c occurs, nor that it is the midpoint average).

Bài tập luyện tập

Let $f(x) = \begin{cases} \dfrac{x^2 - x - 6}{x - 3}, & x < 3 \\ k, & x = 3 \\ -x + 8, & x > 3 \end{cases}$. (a) Simplify the piece for $x < 3$ and find $\lim\limits_{x \to 3^-} f(x)$. (b) Find $\lim\limits_{x \to 3^+} f(x)$. (c) Find the value of k that makes f continuous at $x = 3$. (d) Using this k, is f continuous on all of $\mathbb{R}$? Explain.

Lời giải chi tiết

(a) $x^2 - x - 6 = (x - 3)(x + 2)$, so for $x < 3$, $f(x) = x + 2$. Then $\lim\limits_{x \to 3^-} f(x) = 3 + 2 = 5$.

(b) $\lim\limits_{x \to 3^+} f(x) = \lim\limits_{x \to 3^+} (-x + 8) = -3 + 8 = 5$.

(c) Both one-sided limits equal 5, so the two-sided limit is 5; continuity at $x = 3$ requires $f(3) = k = 5$.

(d) Yes. For $x < 3$, $f(x) = x + 2$ is a polynomial, continuous on $(-\infty, 3)$. For $x > 3$, $f(x) = -x + 8$ is a polynomial, continuous on $(3, \infty)$. At $x = 3$, with $k = 5$, all three continuity conditions hold as shown in (c). Since f is continuous on each piece and matches up seamlessly at the one breakpoint, f is continuous on all of $\mathbb{R}$.

Bài tập luyện tập

Let $g(x) = \begin{cases} \dfrac{x^3-1}{x-1}, & x \neq 1 \\ m, & x = 1 \end{cases}$. (a) Find the value of m that removes the discontinuity at $x = 1$. (b) Using that value of m, use the Intermediate Value Theorem to show that $g(x) = 2$ has a solution in $(0, 1)$, and verify this solution algebraically.

Lời giải chi tiết

(a) Factor: $x^3 - 1 = (x-1)(x^2+x+1)$, so for $x \neq 1$, $g(x) = x^2 + x + 1$. Then $\lim\limits_{x\to 1} g(x) = 1 + 1 + 1 = 3$, so $m = 3$ removes the discontinuity. With this choice, $g(x) = x^2 + x + 1$ for *every* real x — a polynomial, continuous everywhere.
(b) $g(0) = 0 + 0 + 1 = 1$ and $g(1) = 1 + 1 + 1 = 3$. Since g is continuous on $[0, 1]$ and $N = 2$ satisfies $1 < 2 < 3$, the IVT guarantees some $c \in (0, 1)$ with $g(c) = 2$. *Verification:* solve $x^2 + x + 1 = 2 \Rightarrow x^2 + x - 1 = 0$, giving $x = \dfrac{-1 \pm \sqrt{5}}{2}$. The positive root $x = \dfrac{-1 + \sqrt{5}}{2} \approx 0.618$ indeed lies in $(0, 1)$, confirming the IVT's guarantee.

Differentiation: Definition and Fundamental Properties

Mục tiêu Unit: Định nghĩa tốc độ thay đổi trung bình (thương sai phân, hệ số góc cát tuyến) và tốc độ thay đổi tức thời (đạo hàm tại một điểm, hệ số góc tiếp tuyến) thông qua giới hạn; các ký hiệu đạo hàm (Leibniz, Lagrange, Newton) và cách ước lượng đạo hàm từ bảng số liệu hoặc đồ thị; mối quan hệ giữa tính khả vi và tính liên tục, cùng bốn dạng điểm không khả vi (góc nhọn, điểm nhọn/cusp, tiếp tuyến đứng, gián đoạn); quy tắc lũy thừa (kể cả số mũ âm, phân số); quy tắc tổng–hiệu–nhân hằng số; đạo hàm của e^x và $\ln x$; chứng minh đạo hàm $\sin x$, $\cos x$ từ định nghĩa giới hạn; quy tắc tích (product rule) và quy tắc thương (quotient rule); đạo hàm của $\tan x$, $\cot x$, $\sec x$, $\csc x$.

2.1 Average and Instantaneous Rates of Change

2.1.1 The difference quotient and secant lines

Suppose f is a function defined on an interval containing two distinct points a and b. The **average rate of change** of f over $[a, b]$ measures how much the output changes per unit of input, on average, across that interval:

$$\text{average rate of change} = \frac{f(b) - f(a)}{b - a}.$$

This ratio is called the **difference quotient**. Geometrically, the two points $(a, f(a))$ and $(b, f(b))$ both lie on the graph of f, and the line joining them is called a **secant line** (from the Latin *secare*, "to cut" — it cuts through the curve at two points). The difference quotient $\frac{f(b) - f(a)}{b - a}$ is exactly the slope of that secant line, since slope is "rise over run" and the rise is $f(b) - f(a)$ while the run is $b - a$.

The difference quotient

For any two distinct points a, b in the domain of f,

$$\frac{f(b) - f(a)}{b - a}$$

is simultaneously (i) the **average rate of change** of f on $[a, b]$, and (ii) the **slope of the secant line** through $(a, f(a))$ and $(b, f(b))$. If f represents position as a function of time, this quantity is the **average velocity** on $[a, b]$.

VN **GIẢI THÍCH TIẾNG VIỆT**

Thương sai phân $\frac{f(b) - f(a)}{b - a}$ chính là **tốc độ thay đổi trung bình** của f trên đoạn $[a, b]$, đồng thời cũng là **hệ số góc của đường thẳng cát tuyến** nối hai điểm $(a, f(a))$ và $(b, f(b))$ trên đồ thị. Nếu f là hàm vị trí theo thời gian, đại lượng này chính là **vận tốc trung bình**. Ghi nhớ: "trung bình" ở đây có nghĩa là ta chỉ nhìn vào hai điểm đầu–cuối của khoảng, không quan tâm hàm số biến thiên như thế nào ở giữa khoảng đó.

Ví dụ mẫu · Average rate of change as a secant slope

Let $f(x) = x^2 - 2x + 3$. Find the average rate of change of f on $[1, 4]$, and give the equation of the secant line through the corresponding two points.

Compute the two function values: $f(1) = 1 - 2 + 3 = 2$ and $f(4) = 16 - 8 + 3 = 11$.

$$\text{average rate of change} = \frac{f(4) - f(1)}{4 - 1} = \frac{11 - 2}{3} = \frac{9}{3} = 3.$$

The secant line passes through $(1, 2)$ with slope 3: $y - 2 = 3(x - 1)$, i.e. $y = 3x - 1$. This line crosses the parabola at exactly the two points $x = 1$ and $x = 4$ and nowhere else between them.

2.1.2 From secant to tangent: the derivative at a point

The average rate of change depends on the choice of *two* points, but many questions in science and engineering — "how fast is this car going *right now*?", "how fast is the tumor growing *at this instant*?" — ask about a rate of change at a single instant. To capture this, fix the first point at a and let the second point slide toward a. Write the second point as $a + h$, where h is a small (possibly negative) number representing the horizontal distance between the two points. The difference quotient becomes

$$\frac{f(a + h) - f(a)}{h},$$

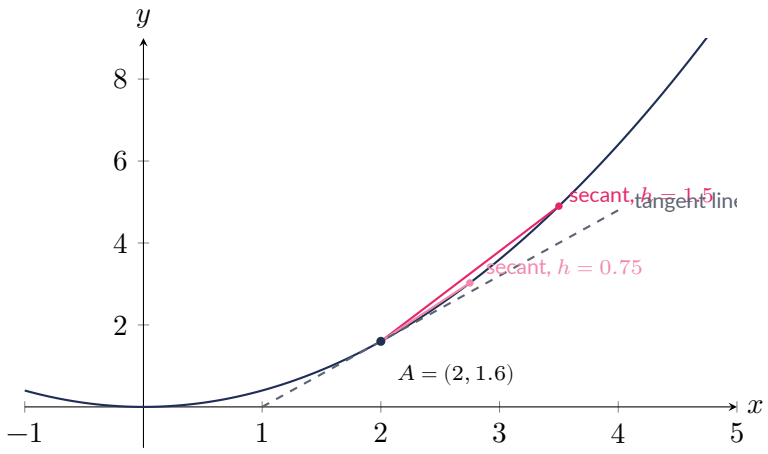
and as $h \to 0$ this secant line's slope approaches the slope of a limiting line called the **tangent line** at $(a, f(a))$ — the line that just "touches" the curve at that one point and best approximates it locally. This limiting slope is the **derivative of** f **at** a, written $f'(a)$:

Definition of the derivative at a point.

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \qquad \text{(equivalently,} \quad f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}\text{)}.$$

When this limit exists (and is finite), f is said to be **differentiable** at a, and $f'(a)$ is simultaneously the **instantaneous rate of change** of f at a and the **slope of the tangent line** to $y = f(x)$ at $x = a$.

The second form, $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$, is the same limit written with $x = a + h$ (so $h = x - a$ and $h \to 0 \iff x \to a$); the two are algebraically interchangeable, and either may be more convenient depending on the problem.



For $f(x) = 0.4x^2$ at $a = 2$: the secant with $h = 1.5$ has slope $\frac{f(3.5) - f(2)}{1.5} = \frac{4.9 - 1.6}{1.5} = 2.2$; the secant with $h = 0.75$ has slope $\frac{f(2.75) - f(2)}{0.75} = \frac{3.025 - 1.6}{0.75} = 1.9$. As $h \to 0$ these slopes approach the tangent slope $f'(2) = 1.6$.

VN **GIẢI THÍCH TIẾNG VIỆT**

Nếu đạo hàm là "hệ số góc của cát tuyến khi hai điểm tiến sát lại gần nhau", thì về bản chất ta đang lấy **giới hạn** của thương sai phân khi $h \to 0$. Đại lượng $f'(a)$ vừa là **tốc độ thay đổi tức thời** tại $x = a$ (ví dụ: vận tốc tức thời nếu f là vị trí), vừa là **hệ số góc của tiếp tuyến** tại điểm $(a, f(a))$. Hình vẽ minh họa: khi h giảm dần ($h = 1.5 \to h = 0.75 \to \cdots \to 0$), đường cát tuyến "xoay dần" và tiến sát tới đường tiếp tuyến — hệ số góc cát tuyến $2.2 \to 1.9 \to \cdots \to 1.6$.

Ví dụ mẫu · Derivative from the limit definition, then the tangent line

Let $f(x) = x^2$. (a) Use the limit definition to find a general formula for $f'(a)$. (b) Evaluate $f'(3)$. (c) Find the equation of the tangent line to $y = x^2$ at $x = 3$.

(a) By definition,

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{(a+h)^2 - a^2}{h} = \lim_{h\to 0} \frac{a^2 + 2ah + h^2 - a^2}{h} = \lim_{h\to 0} \frac{2ah + h^2}{h}.$$

Factor h out of the numerator and cancel (valid since $h \neq 0$ throughout the limit process):

$$f'(a) = \lim_{h\to 0} \frac{h(2a+h)}{h} = \lim_{h\to 0} (2a + h) = 2a.$$

(b) Substituting $a = 3$: $f'(3) = 2(3) = 6$.

(c) The point of tangency is $(3, f(3)) = (3, 9)$ and the slope is $f'(3) = 6$, so the tangent line is

$$y - 9 = 6(x - 3) \implies y = 6x - 9.$$

2.2 Derivative Notation and Numerical Estimation

2.2.1 Leibniz, Lagrange, and Newton notation

Three notations for the derivative are in common use, and it is essential to read all three fluently.

Lagrange notation: $f'(x)$, read "f prime of x"; higher derivatives $f''(x)$, $f'''(x)$, ...

Leibniz notation: $\dfrac{dy}{dx}$ or $\dfrac{d}{dx}[f(x)]$, emphasizing the derivative as a ratio of infinitesimal changes; the value at $x = a$ is written $\left.\dfrac{dy}{dx}\right|_{x=a}$.

Newton notation: a dot over the variable, $\dot{y}$ or $\dot{s}(t)$, used almost exclusively for derivatives with respect to *time* in physics and mechanics (e.g., if $s(t)$ is position, $\dot{s}(t)$ is velocity).

All three notations denote the exact same object; $f'(a) = \left.\dfrac{dy}{dx}\right|_{x=a} = \dot{y}(a)$ whenever $y = f(x)$. Leibniz notation is especially convenient later for the chain rule and for related-rates problems, because it visually tracks which variable is changing with respect to which.

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Ba ký hiệu đạo hàm phổ biến: **Lagrange** ($f'(x)$ — gọn, hay dùng khi làm việc với hàm số trừu tượng); **Leibniz** ($\dfrac{dy}{dx}$ — hữu ích khi cần thấy rõ biến số nào đang thay đổi theo biến số nào, đặc biệt trong các bài liên quan đến tốc độ liên hệ sau này); **Newton** (dấu chấm trên đầu biến, $\dot{y}$ — chỉ dùng cho đạo hàm theo *thời gian* trong vật lý, ví dụ $\dot{s}(t)$ là vận tốc). Cả ba chỉ là ba cách viết khác nhau của cùng một đại lượng.

2.2.2 Estimating derivatives from tables and graphs

When f is known only through a table of sampled values (not a formula), the derivative at an interior point can be estimated using a **symmetric (central) difference quotient** — averaging the secant slope just before and just after the point — which is generally more accurate than a one-sided estimate:

$$f'(a) \approx \frac{f(a+\Delta) - f(a-\Delta)}{2\Delta}.$$

At an endpoint of the table, only a **one-sided difference quotient** (forward or backward) is available.

Ví dụ mẫu · Estimating a derivative from a table of values

A bacterial colony is measured every 2 hours; $P(t)$ is the population in thousands.

t (hours)	0	2	4	6	8
$P(t)$ (thousands)	12	18	29	45	68

Estimate (a) $P'(4)$, (b) $P'(0)$, and (c) $P'(8)$.

(a) $t = 4$ is an interior point, so use the central difference quotient with the neighboring data at $t = 2$ and $t = 6$:

$$P'(4) \approx \frac{P(6) - P(2)}{6 - 2} = \frac{45 - 18}{4} = \frac{27}{4} = 6.75 \text{ thousand/hour.}$$

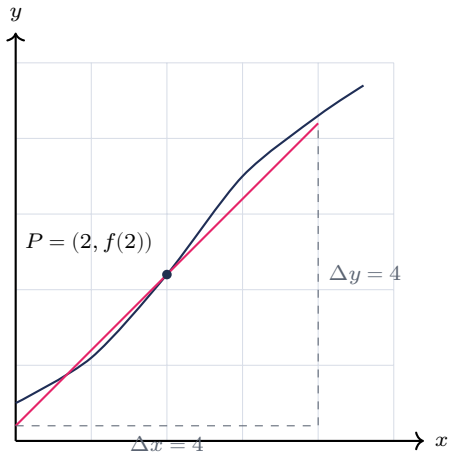
(b) $t = 0$ is the left endpoint of the table, so only a forward difference is available:

$$P'(0) \approx \frac{P(2) - P(0)}{2 - 0} = \frac{18 - 12}{2} = 3 \text{ thousand/hour.}$$

(c) $t = 8$ is the right endpoint, so only a backward difference is available:

$$P'(8) \approx \frac{P(8) - P(6)}{8 - 6} = \frac{68 - 45}{2} = \frac{23}{2} = 11.5 \text{ thousand/hour.}$$

When f is given only as a **graph**, $f'(a)$ can be estimated by sketching the tangent line at $x = a$ as accurately as possible and reading its slope from two clearly identifiable points on that line (ideally where it crosses grid lines).



Estimating $f'(2)$ graphically: the tangent line drawn at P crosses grid points at $(0, 0.2)$ and $(4, 4.2)$, giving slope $\dfrac{\Delta y}{\Delta x} = \dfrac{4.2 - 0.2}{4 - 0} = \dfrac{4}{4} = 1$, so $f'(2) \approx 1$.

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Khi hàm số chỉ được cho dưới dạng **bảng số liệu**, cách ước lượng đạo hàm chính xác nhất tại một điểm *nằm giữa* bảng là dùng **thương sai phân đối xứng (central difference)** — lấy hai điểm cách đều hai bên rồi chia cho tổng khoảng cách; còn tại hai đầu bảng, ta buộc phải dùng thương sai phân **một phía**. Khi hàm số chỉ được cho dưới dạng **đồ thị**, ta ước lượng $f'(a)$ bằng cách vẽ tiếp tuyến tại điểm đó càng chính xác càng tốt, rồi đọc hệ số góc $\dfrac{\Delta y}{\Delta x}$ từ hai điểm rõ ràng trên đường tiếp tuyến đó (tốt nhất là các điểm cắt lưới ô vuông).

2.3 Differentiability and Continuity

2.3.1 Differentiability implies continuity

There is a precise logical relationship between differentiability and continuity, and it runs in only *one* direction.

Theorem: differentiability implies continuity

If f is differentiable at a, then f is continuous at a.

Proof. Suppose $f'(a) = \lim_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ exists as a finite number. To prove continuity at a, it suffices to show $\lim_{x\to a} f(x) = f(a)$, i.e., $\lim_{x\to a} [f(x) - f(a)] = 0$. For $x \neq a$, algebraically rewrite:

$$f(x) - f(a) = \frac{f(x)-f(a)}{x-a} \cdot (x-a).$$

Now take the limit as $x \to a$ of both sides. The right-hand side is a product of two quantities that both have limits as $x \to a$: the first factor tends to $f'(a)$ (by hypothesis), and the second factor $(x-a)$ tends to 0. By the product rule for limits,

$$\lim_{x\to a} [f(x) - f(a)] = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \cdot \lim_{x\to a} (x-a) = f'(a) \cdot 0 = 0.$$

Hence $\lim_{x\to a} f(x) = f(a)$, so f is continuous at a. ■

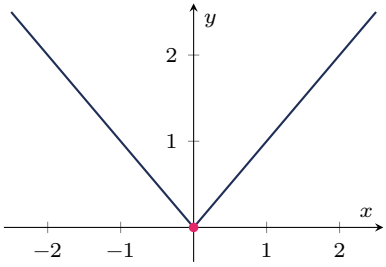
Crucially, the **converse is false**: a function can be continuous at a point yet fail to be differentiable there. Four distinct geometric situations cause this failure.

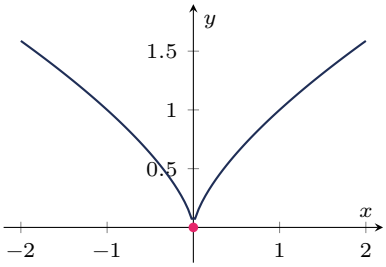
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Chứng minh trên cho thấy: nếu giới hạn $f'(a)$ tồn tại (hữu hạn), thì bắt buộc $\lim_{x\to a} f(x) = f(a)$ — tức là hàm liên tục tại a. Nói cách khác, **khả vi là điều kiện "mạnh" hơn liên tục**: muốn khả vi, hàm phải liên tục *và* còn phải "trơn" (không gấp khúc, không nhọn, không có tiếp tuyến đứng). Chiều ngược lại **KHÔNG** đúng — sẽ có bốn dạng minh họa cụ thể ngay sau đây, mỗi dạng đều là một hàm liên tục nhưng không khả vi tại một điểm.

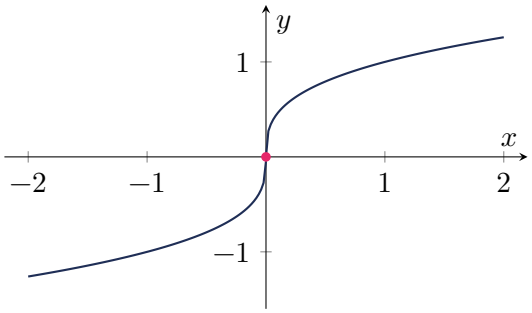
2.3.2 Where derivatives fail to exist

Type	Example (at $x = 0$)	Behavior of $f'(x)$ near 0	Continuous ?
Corner	$f(x) = \lvert x \rvert$	left/right slopes are finite but unequal (-1 vs. $+1$)	Yes
Cusp	$f(x) = x^{2/3}$	slopes $\to -\infty$ from the left, $\to +\infty$ from the right (infinite, opposite signs)	Yes
Vertical tangent	$f(x) = x^{1/3}$	slopes $\to +\infty$ from *both* sides (infinite, same sign)	Yes
Discontinuity	jump at a point	f itself has no limit at the point, so the difference quotient cannot converge	No





Left: $f(x) = |x|$ — a **corner** at $x = 0$ (finite but unequal one-sided slopes). Right: $f(x) = x^{2/3}$ — a **cusp** at $x = 0$ ($f'(x) = \frac{2}{3}x^{-1/3} \to \mp\infty$).



$f(x) = \sqrt[3]{x}$: a **vertical tangent** at $x = 0$. Since $f'(x) = \frac{1}{3}x^{-2/3} \to +\infty$ from *both* sides, the curve is still increasing smoothly through the origin — no corner, just an infinitely steep (vertical) tangent line.

Ví dụ mẫu · Verifying each type of non-differentiability

(a) Corner. For $f(x) = |x-2|+1$, check differentiability at $x = 2$. For $x < 2$, $f(x) = -(x-2)+1 = 3-x$, so $f'(x) = -1$; for $x > 2$, $f(x) = (x-2)+1 = x-1$, so $f'(x) = 1$. The left-hand derivative (-1) and right-hand derivative (1) disagree, so $f'(2)$ does not exist, even though f is continuous at $x = 2$ (both pieces give $f(2) = 1$).

(b) Cusp. For $f(x) = x^{2/3}$, $f'(x) = \frac{2}{3}x^{-1/3}$ for $x \neq 0$. As $x \to 0^+$, $x^{-1/3} \to +\infty$, so $f'(x) \to +\infty$. As $x \to 0^-$, $x^{1/3}$ is negative (e.g. at $x = -0.001$, $x^{1/3} \approx -0.1$), so $x^{-1/3} \to -\infty$, giving $f'(x) \to -\infty$. The one-sided slopes both blow up but with *opposite* signs — a cusp.

(c) Vertical tangent. For $f(x) = x^{1/3}$, $f'(x) = \frac{1}{3}x^{-2/3} = \dfrac{1}{3x^{2/3}}$. Since $x^{2/3} \geq 0$ always and $\to 0^+$ as $x \to 0$ from either side, $f'(x) \to +\infty$ from *both* sides — the same (infinite) sign, so the graph has a genuine vertical tangent rather than a corner.

(d) Discontinuity. Let $f(x) = x + 1$ for $x < 1$ and $f(x) = x^2$ for $x \geq 1$. As $x \to 1^-$, $f(x) \to 2$, but $f(1) = 1^2 = 1$. Since $2 \neq 1$, f has a jump discontinuity at $x = 1$; by the contrapositive of

the theorem above, a function that is not continuous at a point cannot be differentiable there.

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Bốn dạng "gãy" khả vi cần phân biệt rõ: **góc nhọn (corner)** — hai hệ số góc trái/phải *hữu hạn* nhưng khác nhau (như $|x|$); **điểm nhọn (cusp)** — hai hệ số góc trái/phải đều tiến ra *vô cực nhưng trái dấu* (như $x^{2/3}$, tạo hình chữ V nhọn); **tiếp tuyến đứng (vertical tangent)** — hai hệ số góc trái/phải đều tiến ra vô cực *cùng dấu* (như $x^{1/3}$, đồ thị vẫn "trơn" và tăng liên tục, chỉ là tiếp tuyến thẳng đứng); và **điểm gián đoạn** — hàm còn chưa liên tục nên chắc chắn không thể khả vi. Phân biệt cusp và tiếp tuyến đứng thường gây nhầm lẫn nhất: cùng là "vô cực" nhưng cusp có *hai dấu khác nhau* còn tiếp tuyến đứng có *cùng dấu*.

(!) Lỗi thường gặp: "Hàm liên tục thì chắc chắn khả vi" là một suy luận **sai chiều**. Định lý chỉ đảm bảo chiều *khả vi* $\Rightarrow$ *liên tục*, không có chiều ngược lại. Hàm $f(x) = |x|$ là ví dụ kinh điển: liên tục tại $x = 0$ (đồ thị không đứt đoạn) nhưng không khả vi tại đó (đồ thị có góc nhọn, hệ số góc trái $= -1 \neq$ hệ số góc phải $= 1$). Khi được yêu cầu chứng minh một hàm KHÔNG khả vi, đừng chỉ chứng minh nó liên tục — điều đó không đủ.

2.4 The Power Rule and Linearity of Differentiation

2.4.1 The Power Rule

Power Rule. For any real number n for which both sides are defined,

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

As special cases, $\frac{d}{dx}[c] = 0$ for any constant c (an $n = 0$-type function with a flat graph — slope 0 everywhere), and $\frac{d}{dx}[x] = 1$ (an $n = 1$-type function whose graph already is a line of slope 1).

Derivation for a positive integer n. Using the limit definition and the algebraic factorization $x^n - a^n = (x - a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + a^{n-1})$ (a sum of exactly n terms):

$$f'(a) = \lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{x \to a} (x^{n-1} + x^{n-2}a + \cdots + a^{n-1}).$$

As $x \to a$, every one of the n terms in the sum tends to a^{n-1}, so the whole sum tends to $n \cdot a^{n-1}$. Hence $f'(a) = na^{n-1}$, as claimed.

Extension to negative integers. Write $n = -m$ with $m > 0$, so $f(x) = x^{-m} = 1/x^m$. Then

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{\frac{1}{x^m} - \frac{1}{a^m}}{x - a} = \lim_{x \to a} \frac{a^m - x^m}{x^m a^m (x - a)} \\ &= \lim_{x \to a} \left[-\frac{x^m - a^m}{x - a} \cdot \frac{1}{x^m a^m} \right] = -(ma^{m-1}) \cdot \frac{1}{a^{2m}} = -ma^{-m-1}, \end{aligned}$$

using the positive-integer result already proved. This matches nx^{n-1} with $n = -m$: $-mx^{-m-1}$.

Extension to fractional exponents. A fully general proof for arbitrary rational (or irrational) n requires implicit differentiation or logarithmic differentiation, developed in the next chapter; for now, the pattern can be verified directly for specific cases. For instance, with $f(x) = \sqrt{x} = x^{1/2}$,

rationalizing the difference quotient gives

$$f'(a) = \lim_{h\to 0} \frac{\sqrt{a+h}-\sqrt{a}}{h} = \lim_{h\to 0} \frac{(a+h)-a}{h(\sqrt{a+h}+\sqrt{a})} = \lim_{h\to 0} \frac{1}{\sqrt{a+h}+\sqrt{a}} = \frac{1}{2\sqrt{a}},$$

which is exactly $\frac{1}{2}a^{-1/2}$ — matching the power rule with $n = \frac{1}{2}$. The power rule is used freely for every real exponent from this point forward.

VN **GIẢI THÍCH TIẾNG VIỆT**

Quy tắc lũy thừa $\frac{d}{dx}x^n = nx^{n-1}$ áp dụng cho **mọi số mũ thực** n — nguyên dương, nguyên âm, phân số, thậm chí vô tỉ — miễn là cả x^n và x^{n-1} có nghĩa. Với n nguyên dương, ta chứng minh chặt chẽ bằng cách khai triển $x^n - a^n$ thành nhân tử; với n nguyên âm, dùng lại kết quả đó cộng với biến đổi đại số của phân thức; với n phân số, phép chứng minh tổng quát cần đến đạo hàm hàm ẩn (chương sau), nhưng có thể kiểm chứng trực tiếp từng trường hợp cụ thể — ví dụ $\sqrt{x} = x^{1/2}$ bằng cách nhân liên hợp. Điểm mấu chốt: quy tắc lũy thừa chỉ áp dụng khi **biến số nằm ở cơ số** (như x^n), không áp dụng cho hàm mũ có cơ số hằng số (như 2^x) — hai dạng này có công thức đạo hàm hoàn toàn khác nhau.

Ví dụ mẫu · Power rule with negative and fractional exponents

Differentiate each function.

(a) $f(x) = x^5$ (b) $f(x) = x^{-3}$ (c) $f(x) = x^{2/5}$ (d) $f(x) = \sqrt[3]{x^2}$ (e) $f(x) = \dfrac{1}{x^4}$

(a) $f'(x) = 5x^4$.

(b) $f'(x) = -3x^{-4} = -\dfrac{3}{x^4}$.

(c) $f'(x) = \dfrac{2}{5}x^{2/5-1} = \dfrac{2}{5}x^{-3/5}$.

(d) Rewrite $\sqrt[3]{x^2} = x^{2/3}$ first: $f'(x) = \dfrac{2}{3}x^{2/3-1} = \dfrac{2}{3}x^{-1/3}$.

(e) Rewrite $\dfrac{1}{x^4} = x^{-4}$ first: $f'(x) = -4x^{-5} = -\dfrac{4}{x^5}$.

(!) Lỗi thường gặp: Không được nhầm x^n (cơ số biến, số mũ hằng — dùng quy tắc lũy thừa) với a^x (cơ số hằng, số mũ biến — hàm số mũ, có công thức đạo hàm khác hẳn). Ví dụ: $\frac{d}{dx}(2^x) \neq x\cdot 2^{x-1}$ — áp dụng quy tắc lũy thừa vào 2^x là **sai**, vì biến số x nằm ở số mũ chứ không phải cơ số. (Đạo hàm đúng của a^x cần kỹ thuật khác, sẽ gặp sau khi học đạo hàm hàm hợp.) Luôn hỏi: "biến số x đang nằm ở cơ số hay số mũ?" trước khi chọn công thức.

2.4.2 Sum, difference, and constant multiple rules

Differentiation is **linear**: it distributes over addition/subtraction and pulls constant factors outside.

Linearity of differentiation. If f and g are differentiable and c is a constant,

$$\big[f(x) \pm g(x)\big]' = f'(x) \pm g'(x), \qquad \big[c\,f(x)\big]' = c\,f'(x).$$

Proof (sum rule). By the limit definition,

$$\begin{aligned}\frac{d}{dx}[f(x)+g(x)] &= \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h}\\ &= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right].\end{aligned}$$

Since both limits on the right exist (as $f'(x)$ and $g'(x)$), the limit of the sum equals the sum of the limits: $f'(x)+g'(x)$. The difference rule follows identically with a minus sign, and for the constant multiple rule,

$$\frac{d}{dx}[c\,f(x)] = \lim_{h\to 0}\frac{c\,f(x+h)-c\,f(x)}{h} = c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = c\,f'(x). \qquad \blacksquare$$

Combined with the power rule, linearity lets any polynomial be differentiated term by term.

VN **GIẢI THÍCH TIẾNG VIỆT**

Đạo hàm có tính **tuyến tính**: đạo hàm của một tổng bằng tổng các đạo hàm, đạo hàm của một hiệu bằng hiệu các đạo hàm, và hằng số nhân có thể "kéo ra ngoài" phép lấy đạo hàm. Nhờ tính chất này, kết hợp với quy tắc lũy thừa, ta có thể lấy đạo hàm của **bất kỳ đa thức nào** chỉ bằng cách lấy đạo hàm từng số hạng một — đây là kỹ năng nền tảng được dùng lặp lại xuyên suốt cả môn học.

Ví dụ mẫu · Differentiating a polynomial term by term

Differentiate $p(x)=3x^4-5x^3+2x^2-7x+9$, then evaluate $p'(1)$.

Apply linearity and the power rule to each term:

$$p'(x)=3(4x^3)-5(3x^2)+2(2x)-7(1)+0=12x^3-15x^2+4x-7.$$

Then $p'(1)=12-15+4-7=-6$.

Ví dụ mẫu · Where the tangent line is horizontal

Find all points where the tangent line to $f(x)=x^3-3x^2-9x+2$ is horizontal.

A horizontal tangent occurs exactly where $f'(x)=0$. Differentiate:

$$f'(x)=3x^2-6x-9=3(x^2-2x-3)=3(x-3)(x+1).$$

Setting $f'(x)=0$ gives $x=3$ or $x=-1$. Compute the corresponding y-values:

$$f(3)=27-27-27+2=-25, \qquad f(-1)=-1-3+9+2=7.$$

The tangent line is horizontal at $(3,-25)$ and at $(-1,7)$.

2.5 Derivatives of Exponential and Logarithmic Functions

The number $e\approx 2.71828$ can be characterized as the unique base for which the exponential function's growth rate exactly equals its own value; formally, e is defined so that

$$\lim_{h\to 0}\frac{e^h-1}{h}=1 \qquad \left(\text{equivalently } e=\lim_{h\to 0}(1+h)^{1/h}\right).$$

$$\frac{d}{dx}[e^x] = e^x, \qquad \frac{d}{dx}[\ln x] = \frac{1}{x} \ (x > 0).$$

Justification for e^x. By the limit definition,

$$\frac{d}{dx}[e^x] = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x(e^h - 1)}{h} = e^x \cdot \lim_{h\to 0} \frac{e^h - 1}{h} = e^x \cdot 1 = e^x,$$

where the constant factor e^x was pulled out of the limit (it does not depend on h), and the remaining limit equals 1 by the very definition of e above. This is what makes e^x special among exponential functions: it is literally its own derivative.

Justification for $\ln x$. By the limit definition and the logarithm law $\ln(x+h) - \ln x = \ln(1 + \frac{h}{x})$,

$$\frac{d}{dx}[\ln x] = \lim_{h\to 0} \frac{\ln(x+h) - \ln x}{h} = \lim_{h\to 0} \frac{1}{x} \cdot \frac{\ln\left(1 + \frac{h}{x}\right)}{h/x}.$$

Substitute $u = h/x$ (so $u \to 0$ as $h \to 0$, for fixed $x \neq 0$):

$$= \frac{1}{x} \lim_{u\to 0} \frac{\ln(1+u)}{u} = \frac{1}{x} \cdot 1 = \frac{1}{x},$$

using the standard fact $\lim_{u\to 0} \frac{\ln(1+u)}{u} = 1$, which is equivalent (by continuity of $\ln$) to the defining limit $e = \lim_{u\to 0}(1+u)^{1/u}$.

VN **GIẢI THÍCH TIẾNG VIỆT**

Số e được định nghĩa chính xác là cơ số khiến $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$ — nói cách khác, e^x là hàm mũ **duy nhất** mà tốc độ tăng trưởng tại mọi điểm đúng bằng giá trị của chính nó tại điểm đó ($(e^x)' = e^x$). Đây là lý do e^x xuất hiện tự nhiên trong mọi mô hình tăng trưởng/phân rã mũ. Đạo hàm của $\ln x$ ($= 1/x$) có thể suy ra trực tiếp từ định nghĩa giới hạn kết hợp với tính chất logarit, không cần đến quy tắc dây chuyền.

Ví dụ mẫu · Combining exponential and logarithmic derivatives

Let $f(x) = 3e^x - 4\ln x + x^2$. Find $f'(x)$ and evaluate $f'(1)$.

Apply linearity term by term:

$$f'(x) = 3e^x - \frac{4}{x} + 2x.$$

At $x = 1$: $f'(1) = 3e - 4 + 2 = 3e - 2 \approx 6.155$.

2.6 Derivatives of the Sine and Cosine Functions

Two special trigonometric limits are needed to differentiate $\sin x$ and $\cos x$ from first principles:

$$\lim_{h\to 0} \frac{\sin h}{h} = 1, \qquad \lim_{h\to 0} \frac{1 - \cos h}{h} = 0.$$

Derivation of $(\sin x)' = \cos x$. By the limit definition and the angle-addition identity $\sin(x+h) = \sin x \cos h + \cos x \sin h$,

$$\frac{d}{dx}[\sin x] = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}.$$

Group the terms containing $\sin x$ together:

$$= \lim_{h\to 0} \left[-\sin x \cdot \frac{1 - \cos h}{h} + \cos x \cdot \frac{\sin h}{h} \right] = -\sin x \cdot 0 + \cos x \cdot 1 = \cos x.$$

Derivation of $(\cos x)' = -\sin x$. By the same method, using $\cos(x+h) = \cos x\cos h - \sin x\sin h$,

$$\frac{d}{dx}[\cos x] = \lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h}$$
$$= \lim_{h\to 0}\left[-\cos x\cdot\frac{1-\cos h}{h} - \sin x\cdot\frac{\sin h}{h}\right] = -\cos x\cdot 0 - \sin x\cdot 1 = -\sin x.$$

$$\frac{d}{dx}[\sin x] = \cos x, \qquad \frac{d}{dx}[\cos x] = -\sin x.$$

VN **GIẢI THÍCH TIẾNG VIỆT**

Cả hai công thức $(\sin x)' = \cos x$ và $(\cos x)' = -\sin x$ đều được suy ra trực tiếp từ định nghĩa giới hạn của đạo hàm, kết hợp công thức cộng góc lượng giác và **hai giới hạn lượng giác đặc biệt**: $\lim_{h\to 0}\frac{\sin h}{h} = 1$ và $\lim_{h\to 0}\frac{1-\cos h}{h} = 0$. Mẹo ghi nhớ dấu: đạo hàm của sin "giữ nguyên dấu" ra cos; đạo hàm của cos "đổi dấu" ra $-\sin$ — đây là điểm học sinh hay quên nhất (xem phần Lỗi thường gặp bên dưới).

Ví dụ mẫu · Differentiating a sine–cosine combination and its tangent line

Let $g(x) = 2\sin x - 3\cos x + 5$. Find $g'(x)$ and the equation of the tangent line to $y = g(x)$ at $x = 0$.

$$g'(x) = 2\cos x - 3(-\sin x) = 2\cos x + 3\sin x.$$

At $x = 0$: $g(0) = 2(0) - 3(1) + 5 = 2$ and $g'(0) = 2(1) + 3(0) = 2$. The tangent line at $x = 0$ is

$$y - 2 = 2(x - 0) \implies y = 2x + 2.$$

(!) Lỗi thường gặp: Lỗi dấu kinh điển: $\frac{d}{dx}(\cos x) = -\sin x$, **không phải** $+\sin x$. Học sinh thường nhớ nhầm vì sin và cos "trông giống nhau", nhưng chỉ có đạo hàm của $\cos x$ mang dấu trừ. Kiểm tra nhanh bằng đồ thị: tại $x = 0$, $\cos x$ đang *giảm* (đi qua đỉnh cực đại) nên đạo hàm phải âm — khớp với $-\sin(0^+) < 0$ ngay sau 0; nếu nhớ nhầm thành $+\sin x$ thì tại x hơi lớn hơn 0 sẽ cho giá trị dương, mâu thuẫn với việc hàm đang giảm.

2.7 The Product Rule

When differentiating a product of two functions, it is **not** true in general that $(uv)' = u'v'$. The correct rule is:

Product Rule. If $u = u(x)$ and $v = v(x)$ are both differentiable,

$$[u(x)v(x)]' = u'(x)v(x) + u(x)v'(x).$$

In words: derivative of the first, times the second, plus the first, times derivative of the second.

Derivation. By the limit definition, and adding and subtracting the intermediate term $u(x+h)v(x)$ in the numerator:

$$(uv)'(x) = \lim_{h\to 0}\frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$
$$= \lim_{h\to 0}\frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h}.$$

Group and split into two limits:

$$\begin{aligned} &= \lim_{h\to 0}\left[u(x+h)\cdot\frac{v(x+h)-v(x)}{h}\right] + \lim_{h\to 0}\left[v(x)\cdot\frac{u(x+h)-u(x)}{h}\right] \\ &= u(x)v'(x) + v(x)u'(x), \end{aligned}$$

where $u(x+h)\to u(x)$ as $h\to 0$ because u is differentiable, hence continuous (the theorem proved earlier in this chapter). Rearranging gives $u'v + uv'$.

An intuitive picture: think of $u(x)$ and $v(x)$ as the side lengths of a rectangle, so $u(x)v(x)$ is its area. A small change h in x changes u by roughly $u'(x)h$ and v by roughly $v'(x)h$; the resulting change in area is approximately one thin strip of width $u'(x)h$ along one side plus one thin strip of width $v'(x)h$ along the other side (the tiny corner rectangle of area $u'(x)v'(x)h^2$ is negligible as $h\to 0$), giving $\Delta A \approx u'(x)v(x)h + u(x)v'(x)h$ — consistent with the formula after dividing by h.

For a product of **three** factors, apply the two-factor rule twice by treating uvw as $u\cdot(vw)$:

$$(uvw)' = u'(vw) + u(vw)' = u'vw + u(v'w + vw') = u'vw + uv'w + uvw'.$$

VN **GIẢI THÍCH TIẾNG VIỆT**

Quy tắc tích: đạo hàm của *cái thứ nhất*, nhân *cái thứ hai*, cộng *cái thứ nhất* nhân đạo hàm của *cái thứ hai* — **tuyệt đối không** lấy tích của hai đạo hàm riêng lẻ ($u'v'$). Phần chứng minh dùng đúng kỹ thuật "cộng rồi trừ một số hạng trung gian" quen thuộc, và đặc biệt cần dùng lại kết quả "khả vi $\Rightarrow$ liên tục" đã chứng minh ở đầu chương để khẳng định $u(x+h)\to u(x)$. Với tích ba hàm số, chỉ cần áp dụng quy tắc tích hai lần liên tiếp (nhóm hai hàm cuối lại thành một "khối").

Ví dụ mẫu · Product rule with two factors

Differentiate $f(x) = (3x^2+1)(x^3-2x)$ using the product rule, and verify by expanding first.

Let $u = 3x^2+1$ (so $u' = 6x$) and $v = x^3-2x$ (so $v' = 3x^2-2$).

$$\begin{aligned} f'(x) &= u'v + uv' = 6x(x^3-2x) + (3x^2+1)(3x^2-2) \\ &= (6x^4-12x^2) + (9x^4-6x^2+3x^2-2) = 15x^4-15x^2-2. \end{aligned}$$

Check by expanding: $f(x) = 3x^5-6x^3+x^3-2x = 3x^5-5x^3-2x$, so $f'(x) = 15x^4-15x^2-2$ — the two methods agree.

Ví dụ mẫu · Product rule with three factors

Differentiate $g(x) = x\cdot e^x\cdot\cos x$.

Let $u = x$ ($u'=1$), $v = e^x$ ($v' = e^x$), $w = \cos x$ ($w' = -\sin x$). Then

$$g'(x) = u'vw + uv'w + uvw' = e^x\cos x + xe^x\cos x - xe^x\sin x.$$

Factor out e^x:

$$g'(x) = e^x\Big[(1+x)\cos x - x\sin x\Big].$$

(!) Lỗi thường gặp: Lỗi thường gặp nhất khi gặp tích của hai biểu thức chứa biến: học sinh vô tình lấy đạo hàm *từng phần rồi nhân lại* — tức là tính $u'\cdot v'$ thay vì $u'v+uv'$. Kiểm tra nhanh bằng phản ví dụ: với $u = v = x$, tích $uv = x^2$ có đạo hàm đúng là $2x$; nhưng "quy tắc sai"

$u'v' = 1 \cdot 1 = 1 \neq 2x$ — sai hoàn toàn (trừ tại điểm $x = \frac{1}{2}$, một sự trùng hợp số học, không phải quy luật). Bất cứ khi nào thấy **hai biểu thức chứa biến số nhân với nhau**, đó là tín hiệu phải dùng quy tắc tích.

2.8 The Quotient Rule

Quotient Rule. If $u = u(x)$ and $v = v(x)$ are differentiable and $v(x) \neq 0$,

$$\left[\frac{u(x)}{v(x)}\right]' = \frac{u'(x)v(x) - u(x)v'(x)}{\left[v(x)\right]^2}.$$

Derivation from the product rule. Let $q(x) = \dfrac{u(x)}{v(x)}$, so that $u(x) = q(x)v(x)$. Differentiate both sides using the product rule (treating q as an unknown differentiable function for now):

$$u' = q'v + qv'.$$

Solve for q':

$$q' = \frac{u' - qv'}{v} = \frac{u' - \frac{u}{v}v'}{v} = \frac{u'v - uv'}{v^2}$$

(multiplying the numerator's inner fraction by v/v to clear it). This matches the boxed formula. An equivalent derivation works directly from the limit definition using the same add-and-subtract technique as the product rule, and gives the identical result.

Ví dụ mẫu · Quotient rule

Let $h(x) = \dfrac{2x-1}{x^2+3}$. Find $h'(x)$ and evaluate $h'(2)$.

Let $u = 2x - 1$ ($u' = 2$), $v = x^2 + 3$ ($v' = 2x$).

$$h'(x) = \frac{u'v - uv'}{v^2} = \frac{2(x^2+3) - (2x-1)(2x)}{(x^2+3)^2} = \frac{2x^2 + 6 - 4x^2 + 2x}{(x^2+3)^2} = \frac{-2x^2 + 2x + 6}{(x^2+3)^2}.$$

At $x = 2$: numerator $= -2(4) + 2(2) + 6 = -8 + 4 + 6 = 2$; denominator $= (4+3)^2 = 49$. So $h'(2) = \dfrac{2}{49}$.

Ví dụ mẫu · When to rewrite instead of using the full quotient rule

Differentiate $f(x) = \dfrac{1}{x^2+1}$.

Here the numerator is the constant $u = 1$ ($u' = 0$), so the quotient rule simplifies nicely with $v = x^2 + 1$ ($v' = 2x$):

$$f'(x) = \frac{0 \cdot (x^2+1) - 1 \cdot (2x)}{(x^2+1)^2} = -\frac{2x}{(x^2+1)^2}.$$

Notice that $f(x)$ could also be viewed as $u \cdot v^{-1}$ with $u = 1$ and $v = x^2 + 1$; for a *numerator that is not constant*, rewriting $\dfrac{u}{v}$ as $u \cdot v^{-1}$ and differentiating with the product rule requires knowing how to differentiate $v^{-1} = (x^2+1)^{-1}$, which needs the derivative of a *composite* function (an "outer" power applied to an "inner" polynomial). That technique — the chain rule — is developed fully in the next chapter and will make expressions like this one even faster to

differentiate; for now, the quotient rule (or, as here, a simplified case of it) handles every such problem correctly.

(!) Lỗi thường gặp: Hai lỗi phổ biến với quy tắc thương: (1) **sai thứ tự trừ** — viết nhầm thành $uv' - u'v$ thay vì đúng $u'v - uv'$ (đổi thứ tự sẽ đổi dấu toàn bộ kết quả); (2) **quên bình phương mẫu số** — chỉ chia cho $v(x)$ thay vì $[v(x)]^2$. Mẹo ghi nhớ thứ tự: "*đạo hàm tử nhân mẫu, trừ tử nhân đạo hàm mẫu*" (như đọc từ trái sang phải của công thức) — luôn bắt đầu bằng $u'v$, không phải uv'.

2.9 Derivatives of the Other Trigonometric Functions

The remaining four trigonometric derivatives all follow from the quotient rule applied to the identities $\tan x = \dfrac{\sin x}{\cos x}$, $\cot x = \dfrac{\cos x}{\sin x}$, $\sec x = \dfrac{1}{\cos x}$, $\csc x = \dfrac{1}{\sin x}$.

Tangent. With $u = \sin x$ ($u' = \cos x$), $v = \cos x$ ($v' = -\sin x$):

$$(\tan x)' = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x,$$

using the Pythagorean identity $\sin^2 x + \cos^2 x = 1$.

Cotangent. With $u = \cos x$ ($u' = -\sin x$), $v = \sin x$ ($v' = \cos x$):

$$(\cot x)' = \frac{-\sin x \cdot \sin x - \cos x \cdot \cos x}{\sin^2 x} = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x.$$

Secant. With $u = 1$ ($u' = 0$), $v = \cos x$ ($v' = -\sin x$):

$$(\sec x)' = \frac{0 \cdot \cos x - 1 \cdot (-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \tan x \sec x.$$

Cosecant. With $u = 1$ ($u' = 0$), $v = \sin x$ ($v' = \cos x$):

$$(\csc x)' = \frac{0 \cdot \sin x - 1 \cdot \cos x}{\sin^2 x} = -\frac{\cos x}{\sin^2 x} = -\frac{\cos x}{\sin x} \cdot \frac{1}{\sin x} = -\cot x \csc x.$$

Function	Derivative	Function	Derivative
$\sin x$	$\cos x$	$\cot x$	$-\csc^2 x$
$\cos x$	$-\sin x$	$\sec x$	$\sec x \tan x$
$\tan x$	$\sec^2 x$	$\csc x$	$-\csc x \cot x$

VN **GIẢI THÍCH TIẾNG VIỆT**

Cả bốn công thức đạo hàm còn lại (tan, cot, sec, csc) đều được suy ra bằng cách viết chúng dưới dạng thương của $\sin x$ và $\cos x$ rồi áp dụng quy tắc thương — không cần học thuộc lòng máy móc, chỉ cần nhớ **quy tắc thương** và **định nghĩa** của bốn hàm này ($\tan = \sin / \cos$, $\cot = \cos / \sin$, $\sec = 1/\cos$, $\csc = 1/\sin$) là có thể tự suy ra lại bất cứ lúc nào. Mẹo nhỏ: $\tan x$ và $\sec x$ đi cùng nhau trong công thức đạo hàm của nhau ($\sec^2 x$ và $\sec x \tan x$); tương tự $\cot x$ và $\csc x$ luôn xuất hiện có dấu **trừ**.

Ví dụ mẫu · Combining the product rule with the new trig derivatives

Let $f(x) = x\tan x - \sec x$. Find $f'(x)$ and evaluate $f'(0)$.

Apply the product rule to $x\tan x$ (with $u = x$, $u' = 1$, and $v = \tan x$, $v' = \sec^2 x$), then subtract the derivative of $\sec x$:

$$f'(x) = \underbrace{\tan x + x\sec^2 x}_{(x\tan x)'} - \underbrace{\sec x\tan x}_{(\sec x)'}.$$

At $x = 0$: $\tan 0 = 0$ and $\sec 0 = 1$, so $f'(0) = 0 + 0\cdot 1 - 1\cdot 0 = 0$.

Rules established so far.

Power: $\frac{d}{dx}x^n = nx^{n-1}$

Sine/Cosine: $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$

Tan/Cot: $\sec^2 x$, $-\csc^2 x$

Sum/diff/const. mult.: $(f\pm g)' = f'\pm g'$, $(cf)' = cf'$

Product: $(uv)' = u'v + uv'$

Sec/Csc: $\sec x\tan x$, $-\csc x\cot x$

Exp/Log: $\frac{d}{dx}e^x = e^x$, $\frac{d}{dx}\ln x = \frac{1}{x}$

Quotient: $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

2.10 Problem Bank

Bài tập luyện tập

Let $f(x) = x^2 + 3x$. The average rate of change of f on $[2, 5]$ equals:

(A) 10 **(C)** 15

(B) 13 **(D)** 8

Lời giải chi tiết

$f(2) = 4+6 = 10$ and $f(5) = 25+15 = 40$. Average rate of change $= \dfrac{40-10}{5-2} = \dfrac{30}{3} = 10$. **(A).**

Bài tập luyện tập

A particle's position is $s(t) = t^3 - 4t^2 + 2t$ meters. Find its average velocity on $[1, 3]$.

Lời giải chi tiết

$s(1) = 1 - 4 + 2 = -1$ and $s(3) = 27 - 36 + 6 = -3$. Average velocity $= \dfrac{s(3)-s(1)}{3-1} = \dfrac{-3-(-1)}{2} = \dfrac{-2}{2} = -1$ m/s.

Bài tập luyện tập

Using the limit definition of the derivative, $f'(2)$ for $f(x) = x^2 + 1$ equals:

(A) 4 **(C)** 5

(B) 2 **(D)** 1

Lời giải chi tiết

$f'(a) = \lim_{h\to 0} \dfrac{(a+h)^2+1-(a^2+1)}{h} = \lim_{h\to 0} \dfrac{2ah+h^2}{h} = \lim_{h\to 0}(2a+h) = 2a$. At $a=2$: $f'(2) = 4$. **(A)**.

Bài tập luyện tập

Use the limit definition of the derivative to find $f'(x)$ for $f(x) = 3x^2 - 2x$.

Lời giải chi tiết

$$f'(x) = \lim_{h\to 0} \frac{3(x+h)^2 - 2(x+h) - (3x^2-2x)}{h} = \lim_{h\to 0} \frac{6xh + 3h^2 - 2h}{h}$$
$$= \lim_{h\to 0}(6x + 3h - 2) = 6x - 2.$$

(Consistent with the power/linearity rules: $\frac{d}{dx}(3x^2) = 6x$ and $\frac{d}{dx}(-2x) = -2$.)

Bài tập luyện tập

The slope of the tangent line to $y = x^2 - 4x + 1$ at $x = 3$ is:

(A) 2
(B) 6
(C) -2
(D) 4

Lời giải chi tiết

$f'(x) = 2x - 4$, so $f'(3) = 6 - 4 = 2$. **(A)**.

Bài tập luyện tập

Find the equation of the tangent line to $f(x) = x^3 - 2x + 1$ at $x = -1$.

Lời giải chi tiết

$f(-1) = -1 - (-2) + 1 = 2$. $f'(x) = 3x^2 - 2$, so $f'(-1) = 3(1) - 2 = 1$. Tangent line: $y - 2 = 1(x - (-1)) = x + 1$, so $y = x + 3$.

Bài tập luyện tập

Given the table for $g(t)$: $t = 1, 2, 3, 4, 5$ with $g(t) = 5, 9, 15, 23, 33$. The best estimate of $g'(3)$ using a central difference quotient is:

(A) 6
(B) 7
(C) 8
(D) 14

Lời giải chi tiết

$g'(3) \approx \dfrac{g(4) - g(2)}{4 - 2} = \dfrac{23 - 9}{2} = \dfrac{14}{2} = 7$. **(B)**.

Bài tập luyện tập

A smooth curve passes through $(0, 2)$, and the tangent line drawn at the point where $x = 1$ passes through the grid points $(-1, 1)$ and $(3, 5)$. Estimate $f'(1)$ and explain the reasoning.

Lời giải chi tiết

The slope of the drawn tangent line is $\dfrac{\Delta y}{\Delta x} = \dfrac{5-1}{3-(-1)} = \dfrac{4}{4} = 1$. Since the tangent line's slope at the point of tangency equals the derivative there, $f'(1) \approx 1$.

Bài tập luyện tập

Which function fails to be differentiable at $x = 0$ specifically because of a *corner* (finite but unequal one-sided slopes), rather than a cusp or vertical tangent?

(A) $f(x) = x^{1/3}$

(B) $f(x) = |x|$

(C) $f(x) = x^{2/3}$

(D) $f(x) = \sqrt{x}$

Lời giải chi tiết

$|x|$ has left-hand slope -1 and right-hand slope $+1$ at $x = 0$ — both finite but unequal, the defining feature of a corner. ($x^{1/3}$ has a vertical tangent; $x^{2/3}$ has a cusp; $\sqrt{x}$ has a one-sided vertical tangent at its domain endpoint.) **(B)**.

Bài tập luyện tập

Show that $f(x) = |x - 3|$ is continuous at $x = 3$ but not differentiable there.

Lời giải chi tiết

Continuity: $\lim_{x\to 3} |x-3| = 0 = f(3)$, so f is continuous at $x = 3$.
Differentiability: for $x < 3$, $f(x) = 3 - x = -(x-3)$, so $f'(x) = -1$; for $x > 3$, $f(x) = x - 3$, so $f'(x) = 1$. The left-hand derivative (-1) and right-hand derivative (1) are unequal, so $f'(3)$ does not exist — f has a corner at $x = 3$.

Bài tập luyện tập

The derivative of $f(x) = 4x^{3/2} - \dfrac{2}{x^3} + 7$ is:

(A) $6\sqrt{x} + \dfrac{6}{x^4}$

(B) $6\sqrt{x} - \dfrac{6}{x^4}$

(C) $\dfrac{2}{3}\sqrt{x} + \dfrac{6}{x^4}$

(D) $6\sqrt{x} + \dfrac{6}{x^4} + 7$

Lời giải chi tiết

$\dfrac{d}{dx}(4x^{3/2}) = 4 \cdot \frac{3}{2}x^{1/2} = 6\sqrt{x}$. Rewrite $-\dfrac{2}{x^3} = -2x^{-3}$: its derivative is $-2(-3)x^{-4} = 6x^{-4} = \dfrac{6}{x^4}$. The constant 7 differentiates to 0. Sum: $f'(x) = 6\sqrt{x} + \dfrac{6}{x^4}$. **(A)**. (Choice (D) is the classic error of forgetting that a constant term vanishes.)

Bài tập luyện tập

Differentiate $f(x) = \sqrt[3]{x} + \dfrac{1}{\sqrt{x}} - 5x$.

Lời giải chi tiết

Rewrite as $f(x) = x^{1/3} + x^{-1/2} - 5x$. Then

$$f'(x) = \frac{1}{3}x^{-2/3} - \frac{1}{2}x^{-3/2} - 5.$$

Bài tập luyện tập

For $p(x) = 2x^4 - 3x^3 + x - 8$, the value of $p'(1)$ is:

(A) 0 **(C)** -9

(B) 8 **(D)** 1

Lời giải chi tiết

$p'(x) = 8x^3 - 9x^2 + 1$, so $p'(1) = 8 - 9 + 1 = 0$. **(A).**

Bài tập luyện tập

Find all points where the tangent line to $f(x) = 2x^3 - 3x^2 - 12x + 5$ is horizontal.

Lời giải chi tiết

$f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x - 2)(x + 1)$, zero at $x = 2$ and $x = -1$.

$$f(2) = 16 - 12 - 24 + 5 = -15, \qquad f(-1) = -2 - 3 + 12 + 5 = 12.$$

Horizontal tangents occur at $(2, -15)$ and $(-1, 12)$.

Bài tập luyện tập

For $f(x) = 5e^x - 3\ln x$, the value of $f'(1)$ is:

(A) $5e - 3$ **(C)** $5 - 3e$

(B) $5e + 3$ **(D)** $5e - \dfrac{1}{3}$

Lời giải chi tiết

$f'(x) = 5e^x - \dfrac{3}{x}$. At $x = 1$: $f'(1) = 5e - 3$. **(A).**

Bài tập luyện tập

Using the limit definition of the derivative and the fact $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$, show that $\dfrac{d}{dx}[e^x] = e^x$.

Lời giải chi tiết

$$\frac{d}{dx}[e^x] = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^x(e^h-1)}{h} = e^x\lim_{h\to 0}\frac{e^h-1}{h} = e^x\cdot 1 = e^x.$$

The factor e^x does not depend on h, so it can be pulled outside the limit; the remaining limit equals 1 by the defining property of e.

Bài tập luyện tập

For $f(x) = 4\sin x - \cos x$, the value of $f'\left(\frac{\pi}{2}\right)$ is:

(A) 1 **(C)** -1

(B) 4 **(D)** 0

Lời giải chi tiết

$f'(x) = 4\cos x + \sin x$. At $x = \frac{\pi}{2}$: $\cos\frac{\pi}{2} = 0$, $\sin\frac{\pi}{2} = 1$, so $f'(\frac{\pi}{2}) = 0 + 1 = 1$. **(A)**.

Bài tập luyện tập

Using the limit definition and the two special trigonometric limits, derive $\frac{d}{dx}[\cos x] = -\sin x$.

Lời giải chi tiết

Using $\cos(x+h) = \cos x\cos h - \sin x\sin h$:

$$\frac{d}{dx}[\cos x] = \lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h} = \lim_{h\to 0}\left[-\cos x\cdot\frac{1-\cos h}{h} - \sin x\cdot\frac{\sin h}{h}\right].$$

Since $\frac{1-\cos h}{h}\to 0$ and $\frac{\sin h}{h}\to 1$ as $h\to 0$, this limit equals $-\cos x\cdot 0 - \sin x\cdot 1 = -\sin x$.

Bài tập luyện tập

For $f(x) = (x^2+1)(3x-2)$, the value of $f'(2)$ is:

(A) 31 **(C)** 34

(B) 28 **(D)** 22

Lời giải chi tiết

Product rule: $f'(x) = 2x(3x-2) + (x^2+1)(3) = 6x^2 - 4x + 3x^2 + 3 = 9x^2 - 4x + 3$. At $x = 2$: $9(4) - 8 + 3 = 36 - 8 + 3 = 31$. **(A)**. (Check by expansion: $f(x) = 3x^3 - 2x^2 + 3x - 2$, so $f'(x) = 9x^2 - 4x + 3$ — matches.)

Bài tập luyện tập

Differentiate $g(x) = x^2\cdot e^x\cdot\cos x$ and write the result in fully factored form.

Lời giải chi tiết

With $u = x^2$ ($u' = 2x$), $v = e^x$ ($v' = e^x$), $w = \cos x$ ($w' = -\sin x$):

$$g'(x) = u'vw + uv'w + uvw' = 2xe^x \cos x + x^2 e^x \cos x - x^2 e^x \sin x.$$

Factor out xe^x:

$$g'(x) = xe^x\Big[(2+x)\cos x - x\sin x\Big].$$

Bài tập luyện tập

For $f(x) = \dfrac{x-1}{x+1}$, the value of $f'(0)$ is:

(A) 2 **(C)** -2

(B) 1 **(D)** 0.5

Lời giải chi tiết

Quotient rule with $u = x-1$ ($u' = 1$), $v = x+1$ ($v' = 1$): $f'(x) = \dfrac{(x+1)-(x-1)}{(x+1)^2} = \dfrac{2}{(x+1)^2}$.

At $x = 0$: $f'(0) = \dfrac{2}{1} = 2$. **(A)**.

Bài tập luyện tập

Differentiate $h(x) = \dfrac{3x^2-1}{2x+5}$.

Lời giải chi tiết

With $u = 3x^2 - 1$ ($u' = 6x$), $v = 2x + 5$ ($v' = 2$):

$$h'(x) = \frac{6x(2x+5) - (3x^2-1)(2)}{(2x+5)^2} = \frac{12x^2 + 30x - 6x^2 + 2}{(2x+5)^2} = \frac{6x^2 + 30x + 2}{(2x+5)^2}.$$

Bài tập luyện tập

For $f(x) = \tan x + \sec x$, the value of $f'(0)$ is:

(A) 1 **(C)** 2

(B) 0 **(D)** -1

Lời giải chi tiết

$f'(x) = \sec^2 x + \sec x \tan x$. At $x = 0$: $\sec 0 = 1$, $\tan 0 = 0$, so $f'(0) = 1 + 1 \cdot 0 = 1$. **(A)**.

Bài tập luyện tập

Differentiate $f(x) = x\cot x - \csc x$ and evaluate $f'\left(\dfrac{\pi}{2}\right)$.

Lời giải chi tiết

Product rule on $x\cot x$: $\dfrac{d}{dx}(x\cot x) = \cot x + x(-\csc^2 x) = \cot x - x\csc^2 x$. Derivative of $-\csc x$ is $-(-\csc x\cot x) = \csc x\cot x$. So

$$f'(x) = \cot x - x\csc^2 x + \csc x\cot x.$$

At $x = \frac{\pi}{2}$: $\cot\frac{\pi}{2} = 0$, $\csc\frac{\pi}{2} = 1$, so $f'(\frac{\pi}{2}) = 0 - \frac{\pi}{2}(1) + 1(0) = -\dfrac{\pi}{2}$.

Bài tập luyện tập

Let $f(x) = \dfrac{\sin x}{x^2}$ $(x \neq 0)$. Find $f'(x)$ in simplified form, and determine whether f is increasing or decreasing at $x = \pi$.

Lời giải chi tiết

Quotient rule with $u = \sin x$ $(u' = \cos x)$, $v = x^2$ $(v' = 2x)$:

$$f'(x) = \frac{x^2\cos x - 2x\sin x}{x^4} = \frac{x\cos x - 2\sin x}{x^3} \quad \text{(dividing numerator and denominator by } x\text{)}.$$

At $x = \pi$: $\cos\pi = -1$, $\sin\pi = 0$, so $f'(\pi) = \dfrac{\pi(-1) - 0}{\pi^3} = \dfrac{-\pi}{\pi^3} = -\dfrac{1}{\pi^2}$. Since $f'(\pi) < 0$, f is **decreasing** at $x = \pi$.

Bài tập luyện tập

A function is defined piecewise: $f(x) = x^2$ for $x \leq 1$ and $f(x) = 2x - 1$ for $x > 1$. Is f differentiable at $x = 1$?

(A) Yes — both one-sided derivatives equal 2

(B) No — it is a corner since the two pieces are different formulas

(C) No — f is discontinuous at $x = 1$

(D) Cannot be determined without a graph

Lời giải chi tiết

Continuity: $\lim_{x\to 1^-} x^2 = 1$, $\lim_{x\to 1^+}(2x-1) = 1$, and $f(1) = 1^2 = 1$ — all agree, so f is continuous at $x = 1$.
Differentiability: for $x < 1$, $f'(x) = 2x \to 2(1) = 2$; for $x > 1$, $f'(x) = 2$ (constant). Both one-sided derivatives equal 2, so f *is* differentiable at $x = 1$ with $f'(1) = 2$ — the two pieces join smoothly even though they are different formulas. **(A).** (This shows that a piecewise definition does not automatically create a corner; the one-sided derivatives must always be checked directly.)

Bài tập luyện tập

Let $f(x) = x^3e^x$. (a) Find $f'(x)$ using the product rule. (b) Find all x where the tangent line is horizontal, and give the corresponding points.

Lời giải chi tiết

(a) With $u = x^3$ ($u' = 3x^2$), $v = e^x$ ($v' = e^x$):

$$f'(x) = 3x^2e^x + x^3e^x = x^2e^x(3 + x).$$

(b) Since $e^x \neq 0$ for all x, $f'(x) = 0$ exactly when $x^2(3 + x) = 0$, i.e. $x = 0$ or $x = -3$.

$$f(0) = 0, \qquad f(-3) = (-27)e^{-3} = -27e^{-3}.$$

Horizontal tangents occur at $(0, 0)$ and $(-3, -27e^{-3})$.

Bài tập luyện tập

The following table gives values of a differentiable function $r(x)$: $x = 0, 1, 2, 3, 4$ with $r(x) = 2, 5, 9, 14, 20$. (a) Estimate $r'(2)$ using a central difference quotient. (b) Estimate $r'(0)$ using the best available one-sided difference quotient.

Lời giải chi tiết

(a) $r'(2) \approx \dfrac{r(3) - r(1)}{3 - 1} = \dfrac{14 - 5}{2} = \dfrac{9}{2} = 4.5.$

(b) Since $x = 0$ is the left endpoint, use the forward difference: $r'(0) \approx \dfrac{r(1) - r(0)}{1 - 0} = \dfrac{5 - 2}{1} = 3.$

Bài tập luyện tập

$\lim\limits_{h \to 0} \dfrac{\cos(x + h) - \cos x}{h}$ equals:

(A) $\sin x$

(B) $-\sin x$

(C) $\cos x$

(D) $-\cos x$

Lời giải chi tiết

This limit is exactly the definition of $\dfrac{d}{dx}[\cos x]$, which was derived to equal $-\sin x$. **(B).**

Bài tập luyện tập

Let $f(x) = \dfrac{x^2 + 1}{x}$ ($x \neq 0$). (a) Rewrite $f(x)$ as a sum of power functions and differentiate using the power and sum rules. (b) Verify your answer using the quotient rule. (c) Find the equation of the tangent line to $y = f(x)$ at $x = 1$.

Lời giải chi tiết

(a) $f(x) = \dfrac{x^2}{x} + \dfrac{1}{x} = x + x^{-1}$, so $f'(x) = 1 - x^{-2} = 1 - \dfrac{1}{x^2}$.

(b) Quotient rule with $u = x^2 + 1$ ($u' = 2x$), $v = x$ ($v' = 1$):

$$f'(x) = \frac{2x \cdot x - (x^2 + 1) \cdot 1}{x^2} = \frac{2x^2 - x^2 - 1}{x^2} = \frac{x^2 - 1}{x^2} = 1 - \frac{1}{x^2},$$

which matches part (a).
(c) $f(1) = \dfrac{1+1}{1} = 2$ and $f'(1) = 1 - 1 = 0$. Since the slope is 0, the tangent line is horizontal: $y = 2$.

Differentiation: Composite, Implicit, and Inverse Functions

Mục tiêu Unit: Quy tắc dây chuyền (chain rule) cho hàm hợp một lớp lẫn nhiều lớp, kể cả khi phối hợp với quy tắc tích và quy tắc thương; kỹ thuật đạo hàm hàm ẩn (implicit differentiation) để tìm y' và y'' từ một phương trình không giải được y theo x, gồm cả viết phương trình tiếp tuyến và xác định điểm có tiếp tuyến nằm ngang hoặc thẳng đứng; công thức đạo hàm hàm ngược tổng quát $g'(x) = \dfrac{1}{f'(g(x))}$ và cách áp dụng từ bảng giá trị của f và f'; đạo hàm của ba hàm lượng giác ngược cơ bản arcsin x, arccos x, arctan x; chiến lược lựa chọn đúng quy tắc phù hợp cho một biểu thức bất kỳ; và đạo hàm bậc cao (cấp hai, cấp ba, và quy luật tổng quát) của đa thức, hàm lượng giác, và hàm mũ.

The differentiation rules built up so far — the Power Rule, the Sum/Difference Rule, the Product Rule, and the Quotient Rule — handle functions written explicitly and directly in terms of x. This unit extends that toolkit in four directions: differentiating **composite functions** (one function plugged inside another) using the **Chain Rule**; differentiating relations that mix x and y **implicitly**, where y cannot be isolated as a single formula; differentiating **inverse functions**, both in general and for the standard inverse trigonometric functions; and computing derivatives of **higher order**. These four ideas — chain rule, implicit differentiation, inverse functions, and higher-order derivatives — are the machinery behind nearly every later application of the derivative, from related rates to curve sketching to optimization. Mastering these techniques introduces very little in the way of brand-new differentiation *formulas* — a short reference table holds all of them — so the real skill this unit builds is recognizing, quickly and reliably, which combination of rules a given expression calls for, and executing that combination without dropping a factor along the way.

3.1 The Chain Rule

A **composite function** $y = f(g(x))$ is built in two stages: the inner function g is applied to x first, and then the outer function f is applied to the result $g(x)$. Functions such as $\sin(x^2)$, $(3x-7)^5$, $e^{\cos x}$, and $\sqrt{4-x^2}$ are all composite functions of this kind — none of them is a sum, a plain product, or a simple power of x by itself. The **Chain Rule** is the differentiation rule for exactly this situation:

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x).$$

Read this as *"outer times inner."* Differentiate the outer function f, evaluated at the inner function $g(x)$ (leaving $g(x)$ completely untouched inside), and then multiply the result by the derivative of the inner function, $g'(x)$. This second factor matters because $g(x)$ is not simply x: as x changes, the inner function has its own rate of change, and that rate scales how quickly the whole composite output changes. Omitting this factor — differentiating only the outer layer and stopping — is the single most common error at this level of calculus.

Composite functions of this kind are not a mathematical curiosity — they are the rule rather than the exception whenever real quantities are modeled together. The temperature at a weather balloon's location depends on its altitude, which itself changes with time; the area of a spreading oil slick depends on its radius, which itself changes with time; the volume of a melting snowball

depends on its side length, which itself changes with time. Every one of these is naturally written as one function of a second, time-dependent quantity — exactly the composite-function structure the Chain Rule is built to differentiate, and exactly the structure exploited later when studying related rates.

Writing $y = f(u)$ where $u = g(x)$ makes the same idea transparent in **Leibniz notation**:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

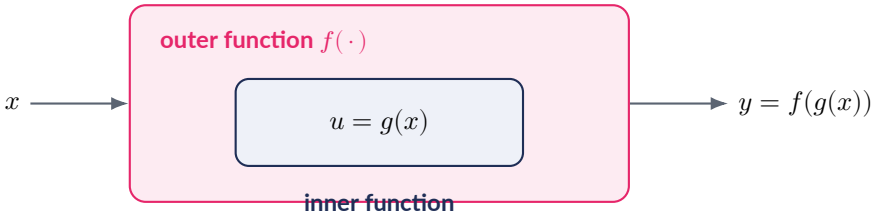
Although $\frac{dy}{du}$ and $\frac{du}{dx}$ are not literally fractions, this notation is a genuinely useful memory device: the "du" factors behave as though they cancel, leaving the overall rate of change of y with respect to x.

Chain Rule

For a composite function $y = f(g(x))$, writing $u = g(x)$:

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x) = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Identify the outer function f and the inner function $u = g(x)$ first; differentiate the outer function at u (leaving u untouched), then multiply by $\frac{du}{dx}$.



The value x is substituted into the inner function g first; the result $u = g(x)$ is then substituted into the outer function f, producing $y = f(g(x))$. Differentiating undoes this in the same order, from the outside in.

VN GIẢI THÍCH TIẾNG VIỆT

Quy tắc dây chuyền: lấy đạo hàm hàm NGOÀI (giữ nguyên hàm trong bên trong nó), rồi NHÂN với đạo hàm hàm TRONG. Luôn tự hỏi "đâu là hàm trong, đâu là hàm ngoài?" trước khi bắt đầu lấy đạo hàm. Ký hiệu Leibniz $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ tuy không phải phép chia phân số thật sự, nhưng là cách ghi nhớ trực quan và hữu ích: các du "triệt tiêu" nhau về mặt hình thức.

Ví dụ mẫu · Basic chain rule

Differentiate $y = (3x^2 - 5)^4$.
The outer function is $(\cdot)^4$ and the inner function is $u = 3x^2 - 5$.

$$\begin{aligned}\frac{dy}{dx} &= 4(3x^2 - 5)^3 \cdot \frac{d}{dx}[3x^2 - 5] \\ &= 4(3x^2 - 5)^3 \cdot 6x \\ &= 24x(3x^2 - 5)^3.\end{aligned}$$

Generalized Power Rule. A special case of the Chain Rule that appears constantly in practice: if $y = [g(x)]^n$ for a differentiable function g and a real exponent n, then

$$\frac{dy}{dx} = n[g(x)]^{n-1} \cdot g'(x).$$

This is simply the ordinary Power Rule, $\frac{d}{dx}[x^n] = nx^{n-1}$, applied with the inner function $g(x)$ standing in for x — the extra factor $g'(x)$ is exactly what the Chain Rule contributes on top of the Power Rule.

The same pattern — "ordinary derivative formula, with an inner function $u = g(x)$ in place of x, times u'" — applies to every basic differentiation formula already known, giving a compact chain-rule reference for the composite forms that appear most often:

$$\frac{d}{dx}[\sin u] = \cos u \cdot u' \qquad \frac{d}{dx}[\cos u] = -\sin u \cdot u'$$
$$\frac{d}{dx}[\tan u] = \sec^2 u \cdot u' \qquad \frac{d}{dx}[e^u] = e^u \cdot u'$$
$$\frac{d}{dx}[\ln u] = \frac{u'}{u} \qquad \frac{d}{dx}[\sqrt{u}] = \frac{u'}{2\sqrt{u}}$$

(In every entry, $u = g(x)$ stands for an arbitrary differentiable inner function, and $u' = g'(x)$.)

3.1.1 Nested (Multi-Layer) Compositions

When a function is built from three or more layers of composition, apply the Chain Rule repeatedly, peeling off exactly one layer at a time, always from the outside in. For a triple composition $y = f(g(h(x)))$:

$$\frac{dy}{dx} = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x).$$

Each layer of composition contributes exactly one multiplicative factor — no layer, however deeply buried, may be skipped. More generally, for a composition of k layers, $y = f_1(f_2(f_3(\cdots f_k(x)\cdots)))$, the derivative is a product of exactly k factors, one contributed by each layer and each evaluated at the appropriate nested input — the same peeling process simply continues for as many layers as the expression actually has.

Ví dụ mẫu · Nested chain rule

Differentiate $y = [\sin(x^2+1)]^3$.
Three layers: outer $(\cdot)^3$, middle $\sin(\cdot)$, inner x^2+1.

$$\begin{aligned}\frac{dy}{dx} &= 3[\sin(x^2+1)]^2 \cdot \cos(x^2+1) \cdot \frac{d}{dx}[x^2+1] \\ &= 3\sin^2(x^2+1)\cos(x^2+1) \cdot 2x \\ &= 6x\sin^2(x^2+1)\cos(x^2+1).\end{aligned}$$

VN **GIẢI THÍCH TIẾNG VIỆT**

Khi có từ ba lớp hàm hợp trở lên, bóc từng lớp một từ NGOÀI vào TRONG, mỗi lớp nhân thêm một đạo hàm tương ứng. Đừng bỏ sót lớp trong cùng — đây là lỗi phổ biến nhất khi hàm hợp có nhiều tầng.

(!) Lỗi thường gặp: Writing $\frac{d}{dx}\sin(3x) = \cos(3x)$ leaves out the derivative of the inner function. The correct derivative is $\cos(3x) \cdot 3 = 3\cos(3x)$. Every layer of a composite function contributes its own factor — never stop differentiating after handling just the outer function.

3.1.2 Combining the Chain Rule with the Product and Quotient Rules

A great many derivatives require more than one rule at once. Recognizing when this happens is mostly a matter of identifying the **outermost** operation, then checking whether any piece underneath it is itself a composite function. If the outermost operation is a **product** of two factors, apply the Product Rule, and apply the Chain Rule to any factor that is itself a composition. If the outermost operation is a **quotient**, apply the Quotient Rule the same way, checking the numerator and the denominator separately for hidden compositions.

In general, if $y = u(x) \cdot v(x)$ where $v(x) = h(k(x))$ is itself a composition, the Product Rule gives $y' = u'v + uv'$, and finding $v' = h'(k(x)) \cdot k'(x)$ is a separate Chain-Rule computation nested inside that larger derivative. The same logic applies when the composition sits in a quotient's numerator, its denominator, or both: differentiate each piece with whatever rule it individually needs, *then* assemble the pieces using the Product or Quotient Rule.

VN **GIẢI THÍCH TIẾNG VIỆT**

Quy tắc tích và quy tắc thương KHÔNG thay thế quy tắc dây chuyền — chúng phối hợp với nhau. Khi áp dụng quy tắc tích/thương, nếu một thừa số (hoặc tử số/mẫu số) là hàm hợp, phải dùng thêm quy tắc dây chuyền để lấy đạo hàm riêng của thừa số đó, rồi mới thay vào công thức tích/thương.

Ví dụ mẫu · Chain rule with the product rule

Differentiate $y = x^3\sin(2x)$.
Product rule with $u = x^3$, $v = \sin(2x)$; note $v' = 2\cos(2x)$ by the chain rule.

$$\begin{aligned}\frac{dy}{dx} &= 3x^2\sin(2x) + x^3 \cdot 2\cos(2x)\\ &= 3x^2\sin(2x) + 2x^3\cos(2x).\end{aligned}$$

Ví dụ mẫu · Chain rule with the quotient rule

Differentiate $y = \dfrac{e^{2x}}{\cos(x^2)}$.
Quotient rule with $u = e^{2x}$, $v = \cos(x^2)$; note $u' = 2e^{2x}$ and $v' = -2x\sin(x^2)$, both by the chain rule.

$$\begin{aligned}\frac{dy}{dx} &= \frac{2e^{2x}\cos(x^2) - e^{2x}(-2x\sin(x^2))}{\cos^2(x^2)}\\ &= \frac{2e^{2x}\cos(x^2) + 2xe^{2x}\sin(x^2)}{\cos^2(x^2)}\\ &= \frac{2e^{2x}[\cos(x^2) + x\sin(x^2)]}{\cos^2(x^2)}.\end{aligned}$$

Dividing each term in the bracket by $\cos^2(x^2)$ gives the equivalent factored form $\frac{dy}{dx} = 2e^{2x}\sec(x^2)[1 + x\tan(x^2)]$.

3.2 Implicit Differentiation

Every function differentiated so far has been given **explicitly**: y isolated on one side of an equation, written directly as a formula in x. Many curves that matter, however, are described by an equation that mixes x and y together in a way that cannot be solved for y as a single formula — or that can only be solved by splitting into multiple awkward branches. The equation of a circle, $x^2 + y^2 = 25$, is one of the simplest examples: solving for y produces two separate branches, $y = \sqrt{25 - x^2}$ and $y = -\sqrt{25 - x^2}$, and messier relations such as $\sin(xy) = x + y^2$ cannot be solved for y in terms of elementary functions at all. Even when an equation *could* in principle be solved for y, doing so is sometimes needlessly painful; implicit differentiation finds $\dfrac{dy}{dx}$ directly from the original equation, without that detour.

Implicit differentiation sidesteps this problem entirely. Instead of solving for y first, treat y as an unspecified differentiable function of x — mentally write $y = y(x)$ — and differentiate *both sides* of the given equation with respect to x, term by term. Every term containing only x differentiates as usual. Every term containing y must be treated as a composite function of x, and therefore requires the Chain Rule, contributing a factor of y' (also written $\dfrac{dy}{dx}$) each time y is differentiated:

$$\frac{d}{dx}[y^n] = n\,y^{n-1}\,y', \qquad \frac{d}{dx}[\sin y] = (\cos y)\,y', \qquad \frac{d}{dx}[e^y] = e^y\,y'.$$

A term containing *both* x and y, such as xy or x^2y^3, additionally requires the Product Rule. After every term has been differentiated, the resulting equation is linear in y': collect every term containing y' on one side, factor y' out, and divide to solve for y' explicitly.

Implicit differentiation, step by step.

1. Differentiate both sides of the equation with respect to x, term by term.
2. Every term containing y picks up a factor of y' (Chain Rule); a term containing both x and y together also needs the Product Rule.
3. Collect every term containing y' on one side of the resulting equation; move every other term to the other side.
4. Factor y' out of the collected terms, then divide to solve for $y' = \dfrac{dy}{dx}$ explicitly, usually leaving an expression in both x and y.

$\dfrac{d}{dx}[y^n]$	$= n\,y^{n-1}y'$	$\dfrac{d}{dx}[xy]$	$= y + xy'$
$\dfrac{d}{dx}[x^2y^3]$	$= 2xy^3 + 3x^2y^2y'$	$\dfrac{d}{dx}[\sin y]$	$= (\cos y)\,y'$
$\dfrac{d}{dx}[e^y]$	$= e^y y'$	$\dfrac{d}{dx}[\ln y]$	$= \dfrac{y'}{y}$
$\dfrac{d}{dx}[\tan y]$	$= \sec^2 y\,y'$	$\dfrac{d}{dx}\left[\dfrac{x}{y}\right]$	$= \dfrac{y - xy'}{y^2}$

VN **GIẢI THÍCH TIẾNG VIỆT**

Đạo hàm hàm ẩn: coi y là một hàm theo x, nên mọi số hạng chứa y khi lấy đạo hàm phải nhân thêm y' (áp dụng quy tắc dây chuyền, vì bản thân y là hàm hợp theo x). Nếu số hạng chứa cả x lẫn y (ví dụ xy, x^2y^3), phải dùng thêm quy tắc tích. Sau khi lấy đạo hàm hai vế, gom tất cả số hạng chứa y' về một phía, đặt y' làm nhân tử chung, rồi chia để giải ra y'.

Ví dụ mẫu · Implicit differentiation

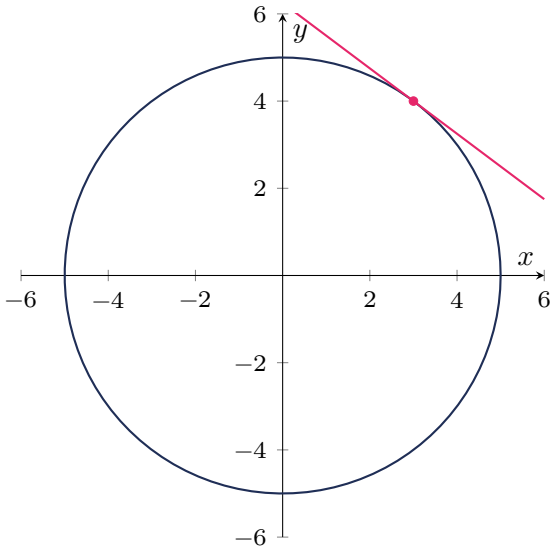
Find y' if $x^2y + y^3 = x + 5$.

Every term on the left involves y, either directly or raised to a power, so each requires the Chain Rule (and the mixed term x^2y additionally requires the Product Rule); the right side is an ordinary polynomial in x alone. Differentiate both sides, using the product rule on x^2y and the chain rule on y^3:

$$\begin{aligned} 2xy + x^2y' + 3y^2y' &= 1 \\ y'(x^2 + 3y^2) &= 1 - 2xy \\ y' &= \frac{1 - 2xy}{x^2 + 3y^2}. \end{aligned}$$

3.2.1 Finding the Tangent Line at a Point

Once y' has been written as a formula in terms of x and y, substituting the coordinates of a specific point that lies on the curve produces a numerical slope — exactly what is needed to write the equation of a tangent line, $y - y_1 = m(x - x_1)$.



Tangent line to $x^2 + y^2 = 25$ at $(3, 4)$.

Ví dụ mẫu · Tangent line to an implicit curve

Find the tangent line to $x^2 + y^2 = 25$ at $(3, 4)$.

Differentiate implicitly:

$$2x + 2yy' = 0 \implies y' = -\frac{x}{y}.$$

At $(3, 4)$: $y' = -\dfrac{3}{4}$. The tangent line is

$$y - 4 = -\frac{3}{4}(x - 3) \implies y = -\frac{3}{4}x + \frac{25}{4}.$$

3.2.2 Horizontal and Vertical Tangent Lines

An implicitly defined curve has a tangent line at essentially every point, including points where an explicit formula $y = f(x)$ would be undefined or multi-valued. Two special cases are worth locating

directly from the formula for y', which for an implicit curve is typically a ratio of two expressions in x and y. A **horizontal tangent** occurs where $y' = 0$ — that is, where the *numerator* of y' equals zero, provided the denominator is nonzero at that same point. A **vertical tangent** occurs where y' is undefined — that is, where the *denominator* of y' equals zero, provided the numerator is nonzero. Because both cases depend on the point actually lying on the curve, always substitute the candidate x- or y-value back into the *original* equation to solve for the matching coordinate, and always check that the other part of the fraction does not also vanish there.

VN **GIẢI THÍCH TIẾNG VIỆT**

Tiếp tuyến NẰM NGANG khi $y' = 0$, tức là TỬ SỐ của biểu thức y' bằng 0 (trong khi mẫu số khác 0 tại điểm đó). Tiếp tuyến THẲNG ĐỨNG khi y' không xác định, tức là MẪU SỐ của y' bằng 0 (trong khi tử số khác 0). Sau khi tìm được giá trị x hoặc y nghi ngờ, luôn thay ngược vào phương trình gốc của đường cong để tìm toạ độ điểm đầy đủ.

Ví dụ mẫu · Horizontal and vertical tangents

Find all points on the ellipse $4x^2 + y^2 = 8$ where the tangent line is horizontal, and all points where it is vertical.

Differentiate implicitly:

$$8x + 2yy' = 0 \implies y' = -\frac{4x}{y}.$$

Horizontal tangents ($y' = 0$, numerator zero, $y \neq 0$): need $-4x = 0 \Rightarrow x = 0$. Substitute into the original equation: $4(0)^2 + y^2 = 8 \Rightarrow y^2 = 8 \Rightarrow y = \pm 2\sqrt{2}$. Since $y \neq 0$ at both points, these are genuine horizontal tangents:

$$(0,\, 2\sqrt{2}) \quad \text{and} \quad (0,\, -2\sqrt{2}).$$

Vertical tangents (y' undefined, denominator zero, numerator nonzero): need $y = 0$. Substitute into the original equation: $4x^2 + 0 = 8 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$. Since the numerator $-4x = \mp 4\sqrt{2} \neq 0$ at both points, these are genuine vertical tangents:

$$(\sqrt{2},\, 0) \quad \text{and} \quad (-\sqrt{2},\, 0).$$

Notice the fourfold symmetry: because the ellipse $4x^2 + y^2 = 8$ is unchanged under $x \to -x$ and under $y \to -y$, the two horizontal-tangent points are reflections of each other across the x-axis, and the two vertical-tangent points are reflections of each other across the y-axis.

3.2.3 Second Derivatives of Implicit Relations

To find the second derivative y'' of an implicitly defined relation, differentiate the expression already found for y' one more time with respect to x, treating y as a function of x exactly as before — any y appearing inside that expression again contributes a factor of y' when differentiated. As soon as a lone y' appears in the resulting expression, substitute the already-known first-derivative formula back in, so the final expression for y'' is written purely in terms of x and y.

This extends naturally to questions of concavity: since $y'' > 0$ corresponds to a curve bending upward and $y'' < 0$ to a curve bending downward, exactly as for an explicit function, the same second-derivative expression found here can be used to decide whether an implicitly defined curve is concave up or concave down at a given point — without ever solving the original equation for y.

Ví dụ mẫu · Implicit second derivative

Continuing the circle $x^2 + y^2 = 25$ with $y' = -\dfrac{x}{y}$, find y'' at $(3, 4)$.

Differentiate $y' = -\dfrac{x}{y}$ using the quotient rule:

$$y'' = -\frac{(1)(y) - x\,y'}{y^2} = -\frac{y - x\left(-\dfrac{x}{y}\right)}{y^2} = -\frac{\dfrac{y^2 + x^2}{y}}{y^2} = -\frac{x^2 + y^2}{y^3}.$$

Since $x^2 + y^2 = 25$ on the curve, this simplifies to $y'' = -\dfrac{25}{y^3}$. At $(3, 4)$:

$$y'' = -\frac{25}{4^3} = -\frac{25}{64}.$$

(!) Lỗi thường gặp: Treating y as a constant while differentiating an equation in x and y is the most common implicit-differentiation error. Since y represents a function of x, every term containing y picks up a factor of y': $\dfrac{d}{dx}[y^2] = 2yy'$ (*not* $2y$), and $\dfrac{d}{dx}[xy] = y + xy'$ (product rule — *not* just y').

VN **GIẢI THÍCH TIẾNG VIỆT**

Để tính đạo hàm cấp hai y'' theo kiểu ẩn, lấy đạo hàm biểu thức y' (đã tìm được) một lần nữa theo x, và thay y' đã biết vào ngay khi nó xuất hiện để rút gọn kết quả theo x và y.

3.3 Derivatives of General Inverse Functions

A function has an inverse only if it is **one-to-one**: no two different inputs may produce the same output, since otherwise there would be no way to decide which input a given output should be sent back to (graphically, this is the horizontal line test). For a differentiable function, one-to-one on an interval is guaranteed whenever f' does not change sign there — that is, whenever f is strictly increasing throughout ($f' > 0$) or strictly decreasing throughout ($f' < 0$) on that interval. Assuming f is one-to-one and differentiable, its inverse $g = f^{-1}$ satisfies, by definition,

$$f(g(x)) = x$$

for every x in the domain of g. This is itself a composite-function identity, so it can be differentiated using the Chain Rule. Differentiating both sides with respect to x — the right-hand side is simply x, whose derivative is 1:

$$f'(g(x)) \cdot g'(x) = 1 \quad \Longrightarrow \quad g'(x) = \frac{1}{f'(g(x))}, \qquad f'(g(x)) \neq 0.$$

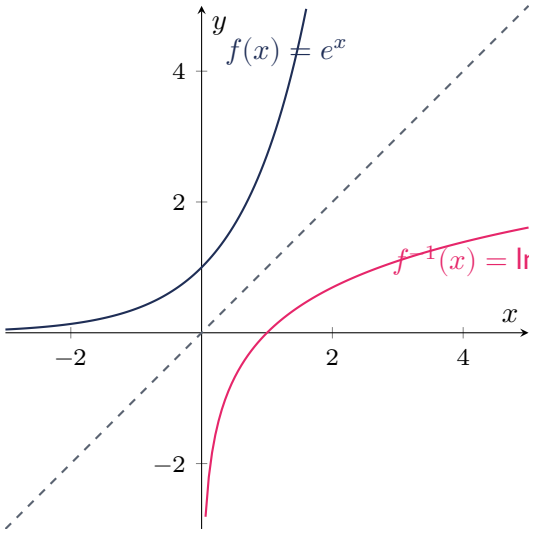
This single formula lets us find the derivative of an inverse function at a point without ever writing down a formula for g itself — all that is required is the value of g at that point (i.e., knowing which input to f produces the desired output) and the value of f' there.

Derivative of an inverse function

If $g = f^{-1}$ is differentiable, then $f(g(x)) = x$ for every x in the domain of g. Differentiating both sides with respect to x:

$$f'(g(x)) \cdot g'(x) = 1 \quad \Longrightarrow \quad g'(x) = \frac{1}{f'(g(x))}, \qquad f'(g(x)) \neq 0.$$

Geometrically, if the point (a, b) lies on the graph of f, then (b, a) lies on the graph of g, and the tangent slopes at these mirrored points are reciprocals of one another.



$f(x) = e^x$ and $f^{-1}(x) = \ln x$ are mirror images of each other across the line $y = x$ (dashed); at mirrored points, their tangent slopes are reciprocals.

VN **GIẢI THÍCH TIẾNG VIỆT**

Nếu g là hàm ngược của f thì đồ thị của g là ảnh phản chiếu của đồ thị f qua đường thẳng $y = x$. Vì vậy $g'(x) = \dfrac{1}{f'(g(x))}$ — hệ số góc của g tại x là NGHỊCH ĐẢO của hệ số góc của f tại điểm tương ứng $g(x)$ (không phải tại x).

The condition $f'(g(x)) \neq 0$ is not a technicality — it identifies exactly the points where the inverse function has a **vertical tangent**. For instance, $f(x) = x^3$ has $f'(x) = 3x^2$, so $f'(0) = 0$; its inverse $g(x) = \sqrt[3]{x}$ indeed has a vertical tangent at $x = 0$, since $g'(x) = \dfrac{1}{3}x^{-2/3}$ is undefined there. Geometrically, a horizontal tangent on the graph of f (slope 0) always mirrors into a vertical tangent on the graph of g (slope undefined) at the corresponding reflected point, echoing the horizontal/vertical tangent analysis already carried out for implicit curves.

To use this formula from a **table of values**, first find $g(x) = f^{-1}(x)$ by locating x in the row of f-values (i.e., finding which input produces that output), then look up f' at that input.

Ví dụ mẫu · Inverse function from a table

x	1	2	3	4
$f(x)$	3	6	10	15
$f'(x)$	2	3	4	5

Let $g = f^{-1}$. Find $g'(6)$.

Since $f(2) = 6$, we have $g(6) = 2$. Then

$$g'(6) = \frac{1}{f'(g(6))} = \frac{1}{f'(2)} = \frac{1}{3}.$$

(!) Lỗi thường gặp: Given $g = f^{-1}$, a frequent error is computing $g'(a) = \dfrac{1}{f'(a)}$ instead of $\dfrac{1}{f'(g(a))}$. The formula requires evaluating f' at $g(a)$ — the *output* of the inverse function — not at the original input a itself. Always find $g(a)$ first (by solving $f(\,\cdot\,) = a$, or reading a table backward), then look up f' there.

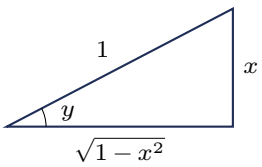
3.4 Derivatives of Inverse Trigonometric Functions

The derivatives of the inverse trigonometric functions can be derived using implicit differentiation. Let $y = \arcsin x$, so that $\sin y = x$ with $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Differentiating both sides with respect to x:

$$\cos y \cdot y' = 1 \quad \Longrightarrow \quad y' = \frac{1}{\cos y}.$$

On $[-\frac{\pi}{2}, \frac{\pi}{2}]$, $\cos y \geq 0$, so $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$. Therefore

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1 - x^2}}.$$



Reference triangle for $y = \arcsin x$: $\sin y = \dfrac{x}{1}$, so the adjacent side is $\sqrt{1 - x^2}$ and $\cos y = \sqrt{1 - x^2}$.

By the same method, let $y = \arccos x$ so that $\cos y = x$ with $y \in [0, \pi]$. Differentiating: $-\sin y \cdot y' = 1$, and since $\sin y \geq 0$ on $[0, \pi]$, $\sin y = \sqrt{1 - x^2}$, giving

$$\frac{d}{dx}[\arccos x] = -\frac{1}{\sqrt{1 - x^2}}.$$

Let $y = \arctan x$ so that $\tan y = x$ with $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Differentiating: $\sec^2 y \cdot y' = 1$, and using the identity $\sec^2 y = 1 + \tan^2 y = 1 + x^2$:

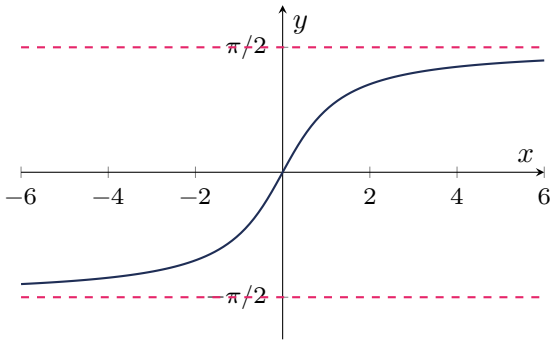
$$\frac{d}{dx}[\arctan x] = \frac{1}{1 + x^2}.$$

The restricted ranges used above — $[-\frac{\pi}{2}, \frac{\pi}{2}]$ for arcsin, $[0, \pi]$ for arccos, and $(-\frac{\pi}{2}, \frac{\pi}{2})$ for arctan — are exactly what make sin, cos, and tan one-to-one and therefore invertible in the first place, and they are also what pin down the sign used to simplify each derivative (e.g., $\cos y \geq 0$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$). On a different branch, the same identities would produce the opposite sign, which is exactly why the standard inverse trigonometric functions are always defined with these particular ranges.

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}} \qquad \text{(domain } -1 < x < 1\text{)}$$
$$\frac{d}{dx}[\arccos x] = -\frac{1}{\sqrt{1-x^2}} \qquad \text{(domain } -1 < x < 1\text{)}$$
$$\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2} \qquad \text{(domain: all real } x\text{)}$$

VN GIẢI THÍCH TIẾNG VIỆT

Ba công thức đạo hàm hàm lượng giác ngược cần thuộc lòng: $\arcsin x$ và $\arccos x$ có CÙNG mẫu số $\sqrt{1-x^2}$ nhưng KHÁC dấu; $\arctan x$ có mẫu số $1+x^2$ và không chứa căn.



The graph of $y = \arctan x$: domain is all real numbers, range is $(-\frac{\pi}{2}, \frac{\pi}{2})$; the horizontal lines $y = \pm\pi/2$ (dashed) are asymptotes that the graph approaches but never reaches.

Ví dụ mẫu · Derivative of arcsin

Differentiate $y = \arcsin(4x)$.
Chain rule with inner function $u = 4x$:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(4x)^2}} \cdot 4 = \frac{4}{\sqrt{1-16x^2}}.$$

Ví dụ mẫu · Derivative of arctan

Differentiate $y = \arctan(3x)$.
Chain rule with inner function $u = 3x$:

$$\frac{dy}{dx} = \frac{1}{1+(3x)^2} \cdot 3 = \frac{3}{1+9x^2}.$$

(!) Lỗi thường gặp: $\arcsin x$ and $\arccos x$ share the same denominator $\sqrt{1-x^2}$ but **opposite signs**: $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ while $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$. This is consistent graphically — $\arcsin x$ is increasing while $\arccos x$ is decreasing on $(-1, 1)$.

(!) Lỗi thường gặp: A common shortcut error is writing $\frac{d}{dx}[\arctan(5x)] = \frac{1}{1+25x^2}$ and stopping there. The formula $(\arctan u)' = \frac{1}{1+u^2}$ is only the *outer* derivative; since the argument

here is $u = 5x$, not x itself, the Chain Rule still requires multiplying by $u' = 5$. The correct derivative is $\dfrac{5}{1+25x^2}$. This omission is the single most common error when combining inverse trig derivatives with the chain rule.

3.5 Selecting Procedures for Differentiation

Most derivatives on an exam require combining more than one rule. Before writing anything down, scan the expression from the outside in and identify the outermost structure.

Decision strategy

Before differentiating, scan the expression from the outside in and ask:

1. Is the outermost operation a **sum or difference**? Differentiate term by term.
2. Is the outermost operation a **product** of two factors? Apply the product rule; check whether either factor itself needs the chain rule.
3. Is the outermost operation a **quotient**? Apply the quotient rule; check whether the numerator or denominator needs the chain rule.
4. Is one function plugged inside another (a **composition**)? Apply the chain rule, peeling one layer at a time from the outside in.
5. Does the equation mix x and y with no way to isolate y, or does y appear more than once? Differentiate **implicitly**, remembering the factor y' on every term containing y.
6. Only substitute specific numeric values **after** the derivative expression has been fully simplified.

The six checks above introduce no new differentiation rule; they are simply a disciplined way of applying the rules already established in this and the previous unit in the correct order, so that no layer of an expression is left undifferentiated and no factor is invented that was never actually part of the original function.

Spot the right tool — four quick diagnoses.

- $y = \dfrac{x^2+1}{\sin(4x)}$: outermost operation is a **quotient** ⇒ Quotient Rule; the denominator $\sin(4x)$ is itself a composition, so its derivative needs the Chain Rule ($4\cos(4x)$) before it is substituted into the quotient formula.
- $y = (x^3 + \tan x)^7$: the outermost operation is a single expression raised to a **power** ⇒ Chain Rule only, with outer function $(\cdot)^7$ and inner function $u = x^3 + \tan x$. There is no separate product or quotient to handle, because the whole sum is packaged inside one set of parentheses.
- $\sqrt{y} + x^3 = y^2$: cannot be solved for y as a single explicit formula ⇒ differentiate **implicitly**. Every term containing y (including $\sqrt{y} = y^{1/2}$) picks up a factor of y'.
- $y = e^{\sin(2x)}\cos x$: outermost operation is a **product** of $e^{\sin(2x)}$ and $\cos x$ ⇒ Product Rule; the first factor is itself a nested composition (outer $e^{(\cdot)}$, inner $\sin(2x)$, innermost $2x$), so differentiating it alone requires two applications of the Chain Rule before the Product Rule can even be written down.

VN GIẢI THÍCH TIẾNG VIỆT

Khi một bài toán kết hợp nhiều quy tắc, hãy xử lý theo đúng THỨ TỰ cấu trúc của biểu thức: nhận diện phép toán NGOÀI CÙNG trước (tổng, tích, thương, hay hàm hợp), áp dụng quy tắc tương ứng, rồi mới xử lý các hàm nằm bên trong.

Ví dụ mẫu · Mixed differentiation strategy

Find y' if $\sin(xy) = x + y^2$.

Reading the equation from the outside in: the left side is a single composition $\sin(\,\cdot\,)$ whose argument xy is itself a product; the right side is a sum whose second term y^2 is itself a power of y. Because y cannot be isolated on one side, the entire equation must be differentiated implicitly, applying the Chain Rule to every piece that involves y. Differentiate both sides:

$$\cos(xy) \cdot \frac{d}{dx}[xy] = 1 + 2yy' \implies \cos(xy)(y + xy') = 1 + 2yy'.$$

Expand and collect the y' terms:

$$\begin{aligned} y\cos(xy) + xy'\cos(xy) &= 1 + 2yy' \\ y'[x\cos(xy) - 2y] &= 1 - y\cos(xy) \\ y' &= \frac{1 - y\cos(xy)}{x\cos(xy) - 2y}. \end{aligned}$$

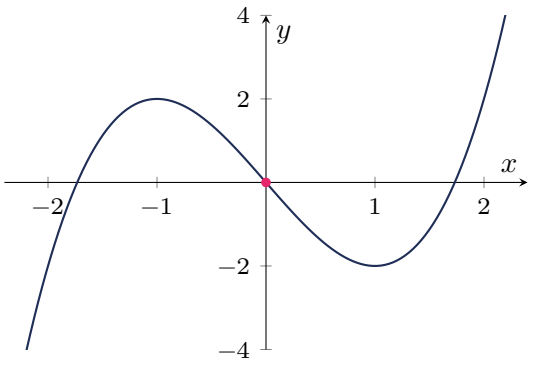
3.6 Higher-Order Derivatives

3.6.1 Notation and the Meaning of Higher-Order Derivatives

Differentiating a function repeatedly produces **higher-order derivatives**. The second derivative is the derivative of the first derivative, the third derivative is the derivative of the second, and so on.

1st derivative:	$f'(x)$	Leibniz:	$\dfrac{dy}{dx}$
2nd derivative:	$f''(x)$	Leibniz:	$\dfrac{d^2y}{dx^2}$
3rd derivative:	$f'''(x)$	Leibniz:	$\dfrac{d^3y}{dx^3}$
n-th derivative:	$f^{(n)}(x)$	Leibniz:	$\dfrac{d^ny}{dx^n}$

The sign of f'' describes how the graph of f curves: $f'' > 0$ corresponds to the graph bending upward and $f'' < 0$ to the graph bending downward, a connection developed further when studying the shape of a graph. Higher-order derivatives also carry direct physical meaning: if $s(t)$ gives the position of an object moving along a line, then $s'(t)$ gives its velocity and $s''(t)$ gives its acceleration, a connection developed in depth in the unit on contextual applications of the derivative.



$y = x^3 - 3x$ bends downward for $x < 0$ (where $y'' < 0$) and upward for $x > 0$ (where $y'' > 0$); the origin is where that behavior switches.

VN **GIẢI THÍCH TIẾNG VIỆT**

Đạo hàm bậc cao là lấy đạo hàm liên tiếp nhiều lần: f'' là đạo hàm của f', f''' là đạo hàm của f'', và cứ tiếp tục như vậy. Dấu của f'' mô tả chiều cong của đồ thị f, một ý sẽ được khai thác sâu hơn khi khảo sát hình dạng đồ thị.

3.6.2 General Patterns for Polynomials, Trig, and Exponential Functions

Three families of functions display especially clean patterns under repeated differentiation, and recognizing them turns what looks like a tedious problem into a quick one. For a **polynomial** of degree m, each differentiation reduces the degree by one, and every derivative of order $m+1$ or higher is identically zero. This happens because a degree-m term x^m becomes mx^{m-1}, then $m(m-1)x^{m-2}$, and so on, until after m differentiations it becomes the constant $m!$ — and differentiating a constant one more time gives 0, after which every further derivative remains 0. For **sine and cosine**, differentiation cycles with period 4:

$$\sin x \xrightarrow{d/dx} \cos x \xrightarrow{d/dx} -\sin x \xrightarrow{d/dx} -\cos x \xrightarrow{d/dx} \sin x \xrightarrow{d/dx} \cdots$$

so the n-th derivative of $\sin x$ (or $\cos x$) depends only on the remainder when n is divided by 4. For the **natural exponential function** e^{kx}, each differentiation multiplies by the constant k, so

$$\frac{d^n}{dx^n}\left[e^{kx}\right] = k^n e^{kx}.$$

Polynomial, degree m	$f^{(n)}(x) = 0$ for every $n > m$ (each derivative lowers the degree by exactly 1)
$\sin x$	cycles $\sin x \to \cos x \to -\sin x \to -\cos x \to \sin x \to \cdots$ (period 4; use n mod 4)
$\cos x$	cycles $\cos x \to -\sin x \to -\cos x \to \sin x \to \cos x \to \cdots$ (period 4; use n mod 4)
e^{kx}	$f^{(n)}(x) = k^n e^{kx}$

Other elementary functions display their own patterns under repeated differentiation. For instance,

$$\frac{d^n}{dx^n}\left[\frac{1}{x}\right] = \frac{(-1)^n\, n!}{x^{n+1}},$$

alternating sign with every derivative while the power of x in the denominator grows by exactly one each time — a pattern easily confirmed by differentiating x^{-1} a few times by hand and tracking the accumulating constant: $-x^{-2}$, then $2x^{-3}$, then $-6x^{-4}$, and so on.

GIẢI THÍCH TIẾNG VIỆT VN

Với hàm lượng giác ($\sin x$, $\cos x$), đạo hàm lặp lại theo chu kỳ 4; muốn tìm đạo hàm cấp n rất lớn, chỉ cần lấy phần dư của n khi chia cho 4 rồi tra chu kỳ. Với hàm mũ e^{kx}, mỗi lần đạo hàm nhân thêm một thừa số k, nên đạo hàm cấp n là $k^n e^{kx}$. Với đa thức bậc m, mọi đạo hàm từ cấp $(m+1)$ trở lên đều bằng 0 — vì mỗi lần đạo hàm làm giảm bậc đi đúng 1.

Ví dụ mẫu · Higher-order and pattern derivatives

(a) Find $f'''(x)$ if $f(x) = x^5 - 3x^4 + 2x$.
Differentiate three times in succession:

$$f'(x) = 5x^4 - 12x^3 + 2, \qquad f''(x) = 20x^3 - 36x^2, \qquad f'''(x) = 60x^2 - 72x.$$

(b) Find $f^{(23)}(x)$ if $f(x) = \sin x$.
Using the period-4 cycle $\sin x \to \cos x \to -\sin x \to -\cos x$, only the remainder of 23 divided by 4 matters: $23 = 4(5) + 3$, so the remainder is 3. The third derivative in the cycle is $-\cos x$, so

$$f^{(23)}(x) = -\cos x.$$

(c) Find $f^{(10)}(x)$ if $f(x) = e^{3x}$.
Each differentiation multiplies by 3, so $f^{(n)}(x) = 3^n e^{3x}$. With $n = 10$ and $3^{10} = 59049$:

$$f^{(10)}(x) = 59049\, e^{3x}.$$

3.7 Problem Bank

Bài tập luyện tập

If $y = (4x+1)^5$, then $\dfrac{dy}{dx} =$

(A) $5(4x+1)^4$
(C) $4(4x+1)^4$
(B) $20(4x+1)^4$
(D) $20(4x+1)^5$

Lời giải chi tiết

Chain rule: $\dfrac{dy}{dx} = 5(4x+1)^4 \cdot 4 = 20(4x+1)^4$. **(B).**

Bài tập luyện tập

$\dfrac{d}{dx}\left[\tan(x^2)\right] =$

(A) $\sec^2(x^2)$
(C) $2x \tan x \sec x$
(B) $2x \sec^2(x^2)$
(D) $\sec^2 x$

Lời giải chi tiết

Outer function $\tan(\cdot)$, inner function $u = x^2$: $\dfrac{d}{dx}\left[\tan(x^2)\right] = \sec^2(x^2) \cdot 2x = 2x \sec^2(x^2)$. **(B).**

Bài tập luyện tập

Find $\dfrac{dy}{dx}$ if $y = \sqrt{4x^3 - x}$.

Lời giải chi tiết

Write $y = (4x^3 - x)^{1/2}$.

$$\frac{dy}{dx} = \frac{1}{2}(4x^3 - x)^{-1/2} \cdot (12x^2 - 1) = \frac{12x^2 - 1}{2\sqrt{4x^3 - x}}.$$

Bài tập luyện tập

Find $\dfrac{d}{dx}[\cos^2(3x)]$.

Lời giải chi tiết

Write $y = [\cos(3x)]^2$. Outer $(\,\cdot\,)^2$, middle $\cos(\,\cdot\,)$, inner $3x$:

$$\frac{dy}{dx} = 2\cos(3x) \cdot (-\sin(3x)) \cdot 3 = -6\sin(3x)\cos(3x).$$

(Equivalently $-3\sin(6x)$ using the double-angle identity $\sin(6x) = 2\sin(3x)\cos(3x)$.)

Bài tập luyện tập

Differentiate: **(a)** $y = x^2\sin(3x)$; **(b)** $y = \dfrac{\cos(3x)}{x^2}$.

Lời giải chi tiết

(a) Product rule, with $\dfrac{d}{dx}[\sin(3x)] = 3\cos(3x)$ by the chain rule:

$$y' = 2x\sin(3x) + 3x^2\cos(3x).$$

(b) Quotient rule, with $\dfrac{d}{dx}[\cos(3x)] = -3\sin(3x)$ by the chain rule:

$$y' = \frac{-3\sin(3x)\cdot x^2 - \cos(3x)\cdot 2x}{x^4} = -\frac{3x\sin(3x) + 2\cos(3x)}{x^3}.$$

Bài tập luyện tập

If $f(x) = \dfrac{e^{2x}}{x+1}$, then $f'(x) =$

(A) $\dfrac{e^{2x}(2x+1)}{(x+1)^2}$

(B) $\dfrac{2e^{2x}}{x+1}$

(C) $\dfrac{e^{2x}(2x-1)}{(x+1)^2}$

(D) $\dfrac{e^{2x}}{(x+1)^2}$

Lời giải chi tiết

Quotient rule, with $u' = 2e^{2x}$ by the chain rule:

$$f'(x) = \frac{2e^{2x}(x+1) - e^{2x}(1)}{(x+1)^2} = \frac{e^{2x}[2(x+1)-1]}{(x+1)^2} = \frac{e^{2x}(2x+1)}{(x+1)^2}.$$

(A).

Bài tập luyện tập

If $x^2 + xy + y^2 = 7$, find $\dfrac{dy}{dx}$ at the point $(1, 2)$.

(A) $-\dfrac{4}{5}$

(B) $\dfrac{4}{5}$

(C) $-\dfrac{5}{4}$

(D) $\dfrac{5}{4}$

Lời giải chi tiết

Differentiate implicitly (product rule on xy):

$$2x + (y + xy') + 2yy' = 0 \implies y'(x+2y) = -(2x+y) \implies y' = -\frac{2x+y}{x+2y}.$$

At $(1,2)$: $y' = -\dfrac{2(1)+2}{1+2(2)} = -\dfrac{4}{5}$. **(A).**

Bài tập luyện tập

Find y' if $y^3 + x^2y = 6$.

Lời giải chi tiết

$$3y^2y' + (2xy + x^2y') = 0 \implies y'(3y^2 + x^2) = -2xy \implies y' = -\frac{2xy}{3y^2+x^2}.$$

Bài tập luyện tập

The tangent line to $x^2 + 4y^2 = 8$ at $(2, 1)$ is

(A) $y = -\dfrac{1}{2}x + 2$

(B) $y = \dfrac{1}{2}x$

(C) $y = -2x + 5$

(D) $y = -\dfrac{1}{2}x + 1$

Lời giải chi tiết

First confirm $(2,1)$ lies on the curve: $2^2 + 4(1)^2 = 4+4 = 8$ ✓. Differentiate: $2x + 8yy' = 0 \Rightarrow y' = -\dfrac{x}{4y}$. At $(2,1)$: $y' = -\dfrac{2}{4} = -\dfrac{1}{2}$. Tangent: $y - 1 = -\dfrac{1}{2}(x-2) \Rightarrow y = -\dfrac{1}{2}x + 2$. **(A).**

Bài tập luyện tập

Find all points on the curve $x^2+xy+y^2=3$ where the tangent line is horizontal, and all points where it is vertical.

Lời giải chi tiết

Differentiate implicitly: $2x+(y+xy')+2yy'=0 \Rightarrow y'(x+2y)=-(2x+y) \Rightarrow y'=-\dfrac{2x+y}{x+2y}$.
Horizontal ($2x+y=0 \Rightarrow y=-2x$, with $x+2y\neq 0$): substitute into $x^2+xy+y^2=3$:

$$x^2+x(-2x)+(-2x)^2=3 \Rightarrow 3x^2=3 \Rightarrow x=\pm 1.$$

This gives $(1,-2)$ and $(-1,2)$; at both points $x+2y=\mp 3\neq 0$, so these are genuine horizontal tangents.
Vertical ($x+2y=0 \Rightarrow x=-2y$, with $2x+y\neq 0$): substitute into $x^2+xy+y^2=3$:

$$(-2y)^2+(-2y)y+y^2=3 \Rightarrow 3y^2=3 \Rightarrow y=\pm 1.$$

This gives $(-2,1)$ and $(2,-1)$; at both points $2x+y=\mp 3\neq 0$, so these are genuine vertical tangents.

Bài tập luyện tập

For $x^2-y^2=16$, find $\dfrac{d^2y}{dx^2}$ in terms of x and y.

(A) $-\dfrac{16}{y^3}$

(C) $-\dfrac{16}{x^3}$

(B) $\dfrac{16}{y^3}$

(D) $\dfrac{y^3}{16}$

Lời giải chi tiết

$2x-2yy'=0 \Rightarrow y'=\dfrac{x}{y}$. Differentiate again with the quotient rule:

$$y''=\frac{(1)y-xy'}{y^2}=\frac{y-\dfrac{x^2}{y}}{y^2}=\frac{y^2-x^2}{y^3}.$$

Since $x^2-y^2=16$, we have $y^2-x^2=-16$, so $y''=-\dfrac{16}{y^3}$. **(A)**.

Bài tập luyện tập

For $xy=6$, find y'' at the point $(2,3)$.

(A) $\dfrac{3}{2}$

(C) $\dfrac{3}{4}$

(B) $-\dfrac{3}{2}$

(D) $-\dfrac{3}{4}$

Lời giải chi tiết

First confirm $(2,3)$ lies on the curve: $2\times 3=6$ ✓. $y+xy'=0\Rightarrow y'=-\dfrac{y}{x}$. Differentiate again:

$$y''=-\frac{y'x-y}{x^2}=\frac{y-y'x}{x^2}.$$

Since $y'x=-y$, the numerator is $y-(-y)=2y$, so $y''=\dfrac{2y}{x^2}$. At $(2,3)$: $y''=\dfrac{2(3)}{4}=\dfrac{3}{2}$. **(A).**

Bài tập luyện tập

Find the slope of the tangent line to $y^2=x^3-3x+1$ at the point $(2,\sqrt{3})$.

Lời giải chi tiết

First confirm $(2,\sqrt{3})$ lies on the curve: $2^3-3(2)+1=8-6+1=3=(\sqrt{3})^2$ ✓. Differentiate: $2yy'=3x^2-3\Rightarrow y'=\dfrac{3x^2-3}{2y}$. At $(2,\sqrt{3})$:

$$y'=\frac{3(4)-3}{2\sqrt{3}}=\frac{9}{2\sqrt{3}}=\frac{9\sqrt{3}}{6}=\frac{3\sqrt{3}}{2}.$$

The tangent line is $y-\sqrt{3}=\dfrac{3\sqrt{3}}{2}(x-2)$.

Bài tập luyện tập

Using the table

x	0	1	2	3
$f(x)$	-1	1	4	9
$f'(x)$	3	4	5	6

let $g=f^{-1}$. Find $g'(4)$.

(A) $\dfrac{1}{5}$ **(C)** $\dfrac{1}{4}$

(B) 5 **(D)** 4

Lời giải chi tiết

Since $f(2)=4$, $g(4)=2$. So $g'(4)=\dfrac{1}{f'(2)}=\dfrac{1}{5}$. **(A).**

Bài tập luyện tập

Using the table

x	2	4	6	8
$f(x)$	5	9	15	23
$f'(x)$	2	3	5	7

let $h = f^{-1}$. Find $h'(15)$.

(A) $\frac{1}{5}$ **(C)** $\frac{1}{3}$

(B) 5 **(D)** 3

Lời giải chi tiết

Since $f(6) = 15$, $h(15) = 6$. So $h'(15) = \dfrac{1}{f'(6)} = \dfrac{1}{5}$. **(A).**

Bài tập luyện tập

Suppose f is differentiable and invertible, $f(3) = 7$, and $f'(3) = \frac{1}{2}$. Find $(f^{-1})'(7)$.

(A) $\frac{1}{2}$ **(C)** 7

(B) 2 **(D)** 3

Lời giải chi tiết

$(f^{-1})'(7) = \dfrac{1}{f'(f^{-1}(7))} = \dfrac{1}{f'(3)} = \dfrac{1}{1/2} = 2$. **(B).**

Bài tập luyện tập

Let $p(x) = x^3 + 2x + 5$ and let $q = p^{-1}$. Find $q'(8)$.

Lời giải chi tiết

Since $p'(x) = 3x^2 + 2 > 0$ for all x, p is increasing and therefore invertible. Solve $p(x) = 8$: $x^3 + 2x - 3 = 0$, and $x = 1$ works since $1 + 2 - 3 = 0$. So $q(8) = 1$.

$$p'(1) = 3(1)^2 + 2 = 5 \quad \Longrightarrow \quad q'(8) = \frac{1}{p'(1)} = \frac{1}{5}.$$

Bài tập luyện tập

Differentiate: **(a)** $y = \arcsin(5x)$; **(b)** $y = \arcsin\left(\dfrac{x}{2}\right)$, and evaluate $y'(0)$ for part (b).

Lời giải chi tiết

(a) $\dfrac{d}{dx}\left[\arcsin(5x)\right] = \dfrac{1}{\sqrt{1-(5x)^2}} \cdot 5 = \dfrac{5}{\sqrt{1-25x^2}}$.

(b)

$$y' = \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}} \cdot \frac{1}{2} = \frac{1/2}{\sqrt{1-x^2/4}}.$$

At $x = 0$: $y'(0) = \dfrac{1/2}{1} = \dfrac{1}{2}$.

Bài tập luyện tập

Find $\dfrac{d}{dx}[\arctan(x^3)]$.

Lời giải chi tiết

$$\frac{d}{dx}[\arctan(x^3)] = \frac{1}{1+(x^3)^2}\cdot 3x^2 = \frac{3x^2}{1+x^6}.$$

Bài tập luyện tập

If $y = \arccos(2x)$, then $y'(0) =$

(A) -2

(B) 2

(C) $-\dfrac{1}{2}$

(D) $\dfrac{1}{2}$

Lời giải chi tiết

$y' = -\dfrac{2}{\sqrt{1-4x^2}}$. At $x=0$: $y'(0) = -\dfrac{2}{\sqrt{1}} = -2$. **(A).**

Bài tập luyện tập

Show that if $y = x\arctan x - \dfrac{1}{2}\ln(1+x^2)$, then $y' = \arctan x$.

Lời giải chi tiết

Differentiate term by term. First term (product rule; the chain rule on $\arctan x$ is trivial since the inner function is x itself):

$$\frac{d}{dx}[x\arctan x] = \arctan x + x\cdot\frac{1}{1+x^2}.$$

Second term (chain rule):

$$\frac{d}{dx}\left[\frac{1}{2}\ln(1+x^2)\right] = \frac{1}{2}\cdot\frac{2x}{1+x^2} = \frac{x}{1+x^2}.$$

Subtracting:

$$y' = \arctan x + \frac{x}{1+x^2} - \frac{x}{1+x^2} = \arctan x.$$

Bài tập luyện tập

If $y = (x^2+1)^3(x-4)^2$, find $y'(4)$.

(A) 0

(B) 17^3

(C) undefined

(D) 1

Lời giải chi tiết

Product rule, with chain rule on $(x^2+1)^3$:

$$y' = 3(x^2+1)^2(2x)(x-4)^2 + (x^2+1)^3 \cdot 2(x-4).$$

Both terms contain a factor of $(x-4)$, so both vanish at $x=4$: $y'(4)=0$. **(A)**.

Bài tập luyện tập

Differentiate $y = \dfrac{\sin(2x)}{x^3}$.

Lời giải chi tiết

Quotient rule, with chain rule on $\sin(2x)$:

$$y' = \frac{2\cos(2x)\cdot x^3 - \sin(2x)\cdot 3x^2}{x^6} = \frac{x^2[2x\cos(2x) - 3\sin(2x)]}{x^6} = \frac{2x\cos(2x) - 3\sin(2x)}{x^4}.$$

Bài tập luyện tập

Find y' at the point $(0,1)$ if $e^{xy} = x + y$.

(A) 0
(B) 1
(C) -1
(D) undefined

Lời giải chi tiết

Check the point: $e^{0\cdot 1} = 1$ and $0+1=1$, so $(0,1)$ is on the curve. Differentiate implicitly (chain rule on e^{xy}, product rule inside):

$$e^{xy}(y + xy') = 1 + y'.$$

At $(0,1)$: $e^0(1 + 0\cdot y') = 1 + y' \Rightarrow 1 = 1 + y' \Rightarrow y' = 0$. **(A)**.

Bài tập luyện tập

Differentiate $y = \ln(\sec x + \tan x)$.

Lời giải chi tiết

Chain rule: $y' = \dfrac{u'}{u}$ where $u = \sec x + \tan x$ and $u' = \sec x \tan x + \sec^2 x$.

$$y' = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x} = \sec x.$$

Bài tập luyện tập

If $f(x) = 3x^4 - 2x^3 + x$, then $f''(x) =$

(A) $36x^2 - 12x$
(C) $36x^2 - 12x + 1$
(B) $12x^2 - 6x$
(D) $24x - 12$

Lời giải chi tiết

$f'(x) = 12x^3 - 6x^2 + 1$, then $f''(x) = 36x^2 - 12x$. **(A).**

Bài tập luyện tập

Find $\dfrac{d^2y}{dx^2}$ if $y = \sin(x^2)$.

Lời giải chi tiết

$y' = 2x\cos(x^2)$ (chain rule). Differentiate again with the product rule, using the chain rule on $\cos(x^2)$:

$$y'' = 2\cos(x^2) + 2x \cdot (-\sin(x^2) \cdot 2x) = 2\cos(x^2) - 4x^2\sin(x^2).$$

Bài tập luyện tập

If $g(x) = e^{2x}$, find $g^{(4)}(x)$, the fourth derivative.

(A) $16e^{2x}$
(C) 2^4xe^{2x}
(B) $8e^{2x}$
(D) $4e^{2x}$

Lời giải chi tiết

Each differentiation multiplies by 2: $g' = 2e^{2x}$, $g'' = 4e^{2x}$, $g''' = 8e^{2x}$, $g^{(4)} = 16e^{2x}$. **(A).**

Bài tập luyện tập

Use implicit differentiation to find y'' at the point $(1, 1)$ on the curve $y^2 = x^3$.

Lời giải chi tiết

First confirm $(1, 1)$ lies on the curve: $1^2 = 1^3$ ✓. Differentiate once: $2yy' = 3x^2 \Rightarrow y' = \dfrac{3x^2}{2y}$.
At $(1, 1)$: $y' = \dfrac{3}{2}$.
Differentiate again, using the quotient rule (equivalently, differentiating $2yy' = 3x^2$ directly with the product rule on the left):

$$2(y')^2 + 2yy'' = 6x \implies y'' = \frac{6x - 2(y')^2}{2y} = \frac{3x}{y} - \frac{(y')^2}{y}.$$

At $(1, 1)$ with $y' = \dfrac{3}{2}$:

$$y'' = 3 - \left(\frac{3}{2}\right)^2 = 3 - \frac{9}{4} = \frac{3}{4}.$$

Bài tập luyện tập

Using the pattern for repeated differentiation, find: **(a)** $f^{(15)}(x)$ if $f(x) = \cos x$; **(b)** $f^{(6)}(x)$ if $f(x) = e^{-2x}$.

Lời giải chi tiết

(a) The cycle for $\cos x$ is $\cos x \to -\sin x \to -\cos x \to \sin x \to \cos x \to \cdots$ (period 4, starting from the first derivative). Since $15 = 4(3) + 3$, the remainder is 3, and the third derivative in the cycle is $\sin x$:

$$f^{(15)}(x) = \sin x.$$

(b) Each differentiation of e^{-2x} multiplies by -2, so $f^{(n)}(x) = (-2)^n e^{-2x}$. With $n = 6$ (even, so the sign is positive) and $(-2)^6 = 64$:

$$f^{(6)}(x) = 64\, e^{-2x}.$$

Contextual Applications of Differentiation

Mục tiêu Unit: Diễn giải ý nghĩa của đạo hàm trong bối cảnh thực tế, kèm đơn vị và câu văn chính xác; phân tích chuyển động thẳng qua vị trí – vận tốc – tốc độ – gia tốc, xác định khi vật tăng tốc, giảm tốc, hay đổi hướng; thiết lập và diễn giải đạo hàm trong các bối cảnh phi chuyển động như kinh tế (chi phí biên, doanh thu biên, lợi nhuận biên), dân số, thể tích, diện tích; nắm vững phương pháp tốc độ liên hệ (related rates) theo quy trình năm bước và áp dụng cho hình cầu giãn nở, thang trượt, bóng đổ, và bể nước hình nón; xấp xỉ giá trị hàm số bằng đường tiếp tuyến (linearization) và vi phân (differentials), đánh giá sai số truyền; sử dụng quy tắc L'Hôpital để tính giới hạn dạng vô định $\frac{0}{0}$ và $\frac{\infty}{\infty}$.

4.1 Interpreting the Derivative in Applied Context

4.1.1 Units of the derivative

Whenever $y = f(x)$ models a real quantity, the derivative $f'(x) = \dfrac{dy}{dx}$ is itself a rate of change, and rates of change always carry **compound units**: units of the output divided by units of the input.

$$\text{units of } f'(x) \;=\; \frac{\text{units of } f(x)}{\text{units of } x}.$$

If $f(x)$ is measured in liters and x in minutes, then $f'(x)$ is measured in liters per minute. If $P(t)$ is a population count (people) and t is measured in years, $P'(t)$ is measured in people per year. The units of f' are never the same as the units of f itself – this is one of the most reliable checks that a derivative has been set up correctly.

The **sign** of $f'(a)$ tells the *direction* of change: positive means the quantity is increasing at $x = a$, negative means it is decreasing. The **magnitude** $|f'(a)|$ tells *how fast*: a large magnitude means rapid change, and a magnitude near 0 means the quantity is nearly stationary (momentarily) at that instant. It is important to remember that $f'(a)$ is an *instantaneous* rate – the rate at which f would change over the next unit of x *if* that instantaneous rate stayed constant, which in general it does not.

VN **GIẢI THÍCH TIẾNG VIỆT**

Đơn vị của đạo hàm $f'(x)$ luôn là **đơn vị của** f **chia cho đơn vị của** x – không bao giờ trùng với đơn vị của chính f. Đây là cách kiểm tra nhanh xem một đạo hàm có được thiết lập đúng hay không. **Dấu** của $f'(a)$ cho biết đại lượng đang tăng hay giảm; **độ lớn** $|f'(a)|$ cho biết tốc độ thay đổi nhanh hay chậm. $f'(a)$ là tốc độ **tức thời** tại đúng thời điểm $x = a$, không phải tốc độ trung bình trên cả một khoảng.

4.1.2 Writing correct interpretation sentences

A correctly written interpretation of $f'(a) = k$ must contain four pieces of information: the identity of the quantity being described, the direction of change (increasing/decreasing, from the sign of k), the numerical rate $|k|$ together with its correct compound units, and the specific input value a (with its own units).

A template for interpretation sentences

"At [input value a, with units], [the quantity described by f] is [increasing / decreasing] at a rate of [$|k|$, with compound units]."

A sentence that reports only the bare number k – with no units, no direction, and no mention of what quantity or instant is being discussed – is not an acceptable interpretation, even if the number itself is correct.

Ví dụ mẫu · Interpreting a derivative value

Let $W(t)$ be the number of gallons of water remaining in a fire truck's tank t minutes after it begins spraying water on a fire, with $W(6) = 450$ and $W'(6) = -25$. Interpret both values in the context of the problem.

$W(6) = 450$ is a *function value*: six minutes after spraying begins, the tank holds 450 gallons of water.

$W'(6) = -25$ is a *derivative value*, with units gallons per minute (gallons, the units of W, divided by minutes, the units of t). Since the sign is negative: at $t = 6$ minutes, the amount of water in the tank is *decreasing* at a rate of 25 gallons per minute.

It is worth distinguishing the **average** rate of change of f over an interval $[a, b]$, given by $\dfrac{f(b) - f(a)}{b - a}$ (the slope of a secant line through $(a, f(a))$ and $(b, f(b))$), from the **instantaneous** rate of change $f'(a)$ (the slope of the tangent line at the single point $x = a$). The average rate summarizes overall behavior across a whole interval and can hide large swings within it; $f'(a)$ describes behavior at one exact instant only. When a problem supplies an explicit formula for f, differentiation rules give $f'(a)$ exactly; when a problem supplies only a table of values, as below, an average rate of change over a nearby interval is the best available *estimate* of $f'(a)$.

Ví dụ mẫu · Estimating a rate from a table of values

The amount of water $W(t)$, in liters, in a drainage tank is recorded every 10 minutes:

t (min)	0	10	20	30	40
$W(t)$ (liters)	500	430	375	335	310

Use the data to estimate $W'(25)$, and write a sentence interpreting your result, including units.

Since $t = 25$ is not one of the recorded times, estimate $W'(25)$ using the average rate of change over the smallest available interval centered at $t = 25$, namely $[20, 30]$:

$$W'(25) \approx \frac{W(30) - W(20)}{30 - 20} = \frac{335 - 375}{10} = \frac{-40}{10} = -4 \text{ L/min.}$$

Interpretation: at $t = 25$ minutes, the amount of water in the tank is decreasing at an approximate rate of 4 liters per minute.

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Khi đề bài cho một **bảng số liệu** rời rạc (không có công thức tường minh), ta ước lượng đạo hàm tại một thời điểm $t = a$ nằm giữa hai giá trị đã cho bằng **tốc độ thay đổi trung bình (average rate of change)** trên khoảng nhỏ nhất chứa a mà bảng cung cấp: $f'(a) \approx \dfrac{f(b) - f(c)}{b - c}$ với $c < a < b$ càng gần a càng tốt. Đây là kỹ thuật xuất hiện thường xuyên trong đề thi AP khi

dữ liệu chỉ có dạng bảng.

4.1.3 Function value versus rate of change

A recurring source of error in applied problems is confusing $f(a)$ (a **state** – how much of the quantity exists at $x = a$) with $f'(a)$ (a **rate** – how fast that quantity is changing at $x = a$). These two numbers answer entirely different questions and, in general, have no numerical relationship to each other: a quantity can be large but decreasing ($f(a)$ large, $f'(a) < 0$), small but increasing rapidly ($f(a)$ small, $f'(a)$ large and positive), or any other combination.

(!) Lỗi thường gặp: A written interpretation that reports a rate of change (units: [output]/[input]) without stating those compound units, or that reports a bare number with no sentence explaining what is increasing/decreasing and at what instant, will not communicate the intended meaning – and on a scored free-response item, will not earn full credit even if the numerical value is correct. Always attach the correct compound units and identify the specific instant being described.

4.2 Straight-Line Motion

4.2.1 Position, velocity, and acceleration

For a particle moving along a straight line (the x-axis or a number line), let $s(t)$ denote its **position** at time t. Two further functions are obtained by successive differentiation:

$$v(t) = s'(t) \quad (\textbf{velocity}), \qquad a(t) = v'(t) = s''(t) \quad (\textbf{acceleration}).$$

Speed is defined as $|v(t)|$ – the magnitude of velocity – and is never negative. Velocity, by contrast, is *signed*: it encodes both how fast the particle is moving *and* which direction it is moving. The particle moves in the positive direction when $v(t) > 0$, in the negative direction when $v(t) < 0$, and is momentarily at rest when $v(t) = 0$.

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$s(t)$ là **vị trí**, $v(t) = s'(t)$ là **vận tốc** (có dấu – dương nghĩa là chuyển động theo chiều dương, âm nghĩa là chuyển động theo chiều âm), và $a(t) = v'(t) = s''(t)$ là **gia tốc**. **Tốc độ (speed)** $= |v(t)|$ luôn không âm – đây là đại lượng đo "độ nhanh chậm" thuần túy, không quan tâm chiều chuyển động. Vật đứng yên tức thời khi $v(t) = 0$.

4.2.2 Speed and the direction of motion

Because speed is $|v(t)|$, the particle "speeding up" means $|v(t)|$ is *increasing*, and "slowing down" means $|v(t)|$ is *decreasing*. Whether the speed is increasing or decreasing at a given instant is determined not by $v(t)$ alone, but by comparing the *signs* of $v(t)$ and $a(t)$ at that instant.

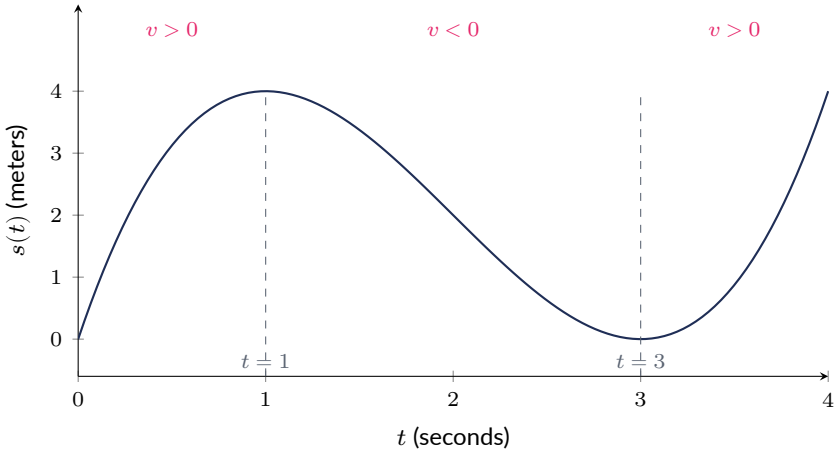
Speeding up versus slowing down: at a given instant t,

- if $v(t)$ and $a(t)$ have the **same sign** (both positive or both negative), the speed $|v(t)|$ is **increasing** – the particle is speeding up;
- if $v(t)$ and $a(t)$ have **opposite signs**, the speed $|v(t)|$ is **decreasing** – the particle is slowing down;
- if $v(t) = 0$ or $a(t) = 0$ at that instant, the particle is (instantaneously) neither speeding up nor slowing down.

(!) Lỗi thường gặp: Velocity and speed are *not* the same quantity, and "the velocity is decreasing" does *not* mean "the particle is slowing down." For example, if $v(t)$ goes from -2 to -5 m/s, the velocity value has decreased ($-5 < -2$), yet the *speed* has increased from 2 to 5 m/s – the particle is moving in the negative direction and speeding up. Never judge speeding up/slowing down from whether $v(t)$ itself is numerically growing or shrinking; always compare the **signs** of $v(t)$ and $a(t)$.

4.2.3 Speeding up, slowing down, and changing direction

A zero of $v(t)$ means the particle is momentarily at rest, but this does *not* automatically mean the particle reverses direction there. The particle genuinely **changes direction** at $t = c$ only if $v(c) = 0$ *and* $v(t)$ actually changes sign as t passes through c (from positive to negative, or negative to positive). If $v(t) = 0$ at $t = c$ but $v(t)$ has the *same* sign immediately before and after c, the particle merely pauses and continues in the same direction.



Position $s(t) = t^3 - 6t^2 + 9t$. The curve rises where $v(t) > 0$ (moving right) and falls where $v(t) < 0$ (moving left). Direction reverses exactly where v changes sign, at $t = 1$ and $t = 3$.

Ví dụ mẫu · Full motion analysis of a cubic position function

A particle moves along a line with position $s(t) = t^3 - 6t^2 + 9t$ (meters, t in seconds) for $0 \le t \le 4$, graphed above. Find when the particle is at rest, describe its direction of motion, determine when it speeds up or slows down, and state the times at which it changes direction.

Velocity: $v(t) = s'(t) = 3t^2 - 12t + 9 = 3(t-1)(t-3)$. Setting $v(t) = 0$ gives $t = 1$ and $t = 3$: the particle is at rest at these two instants.

Direction: testing a value in each sub-interval, $v(t) > 0$ on $(0, 1)$, $v(t) < 0$ on $(1, 3)$, and $v(t) > 0$ on $(3, 4)$. So the particle moves *right* on $(0, 1)$, *left* on $(1, 3)$, and *right* again on $(3, 4)$.

Acceleration: $a(t) = v'(t) = 6t - 12 = 6(t-2)$, so $a(t) < 0$ on $(0, 2)$ and $a(t) > 0$ on $(2, 4)$.

Speeding up / slowing down – compare the signs of v and a on each sub-interval:

Interval	sign of v	sign of a	same/opposite	conclusion
$(0, 1)$	$+$	$-$	opposite	slowing down
$(1, 2)$	$-$	$-$	same	speeding up
$(2, 3)$	$-$	$+$	opposite	slowing down
$(3, 4)$	$+$	$+$	same	speeding up

Change of direction: since $v(t)$ actually changes sign at $t = 1$ (from $+$ to $-$) and again at $t = 3$ (from $-$ to $+$), the particle reverses direction at **both** $t = 1$ **and** $t = 3$.

Ví dụ mẫu · Motion given by a velocity function directly

A particle moves along the x-axis with velocity $v(t) = t^2 - 6t + 5$ (m/s) for $0 \le t \le 6$. Determine whether the particle is speeding up or slowing down at $t = 4$, and find all times in $[0, 6]$ at which the particle changes direction.

Factor: $v(t) = (t-1)(t-5)$, so $v(t) = 0$ at $t = 1$ and $t = 5$. Testing signs: $v > 0$ on $[0, 1)$, $v < 0$ on $(1, 5)$, $v > 0$ on $(5, 6]$. Since v genuinely changes sign at both zeros, the particle **changes direction at** $t = 1$ **and at** $t = 5$.

Acceleration: $a(t) = v'(t) = 2t - 6 = 2(t-3)$, so $a(t) < 0$ on $[0, 3)$ and $a(t) > 0$ on $(3, 6]$.

At $t = 4$: $v(4) = 16 - 24 + 5 = -3$ (so $v < 0$), and $a(4) = 2(4) - 3 \cdot 2 = 2(4) - 6 = 2$ (so $a > 0$). Since $v(4) < 0$ and $a(4) > 0$ have **opposite signs**, the particle is **slowing down** at $t = 4$ (speed at that instant is $|v(4)| = 3$ m/s, and that speed is decreasing).

Note that $a(t) = 0$ at $t = 3$, but $v(3) = 9 - 18 + 5 = -4 \neq 0$ there – the acceleration vanishing does not cause a direction change; only a sign change in $v(t)$ does, and v does not change sign at $t = 3$.

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Đừng lẫn hai loại "điểm đặc biệt": $v(t) = 0$ (vật **đứng yên tức thời** – chỉ đổi hướng nếu v thực sự đổi dấu quanh đó) và $a(t) = 0$ (chỉ là ranh giới giữa hai kiểu tăng/giảm tốc, **không** liên quan đến việc đổi hướng). Ở Ví dụ mẫu thứ hai, $a(3) = 0$ nhưng vật không hề đổi hướng tại $t = 3$ vì $v(3) = -4 \neq 0$ – gia tốc bằng 0 chỉ đánh dấu ranh giới chuyển từ "giảm tốc" sang "tăng tốc" (hoặc ngược lại), không phải điểm đổi chiều chuyển động.

4.3 Rates of Change Beyond Motion

4.3.1 Marginal cost, revenue, and profit

The derivative-as-rate-of-change idea applies far beyond motion. In economics, if $C(x)$ is the total cost (in dollars) to produce x units of a good, the derivative $C'(x)$ is called the **marginal cost**: the approximate cost, in dollars per unit, of producing one additional unit beyond the current production level x. Likewise, if $R(x)$ is total revenue from selling x units, $R'(x)$ is the **marginal revenue**; and if $P(x) = R(x) - C(x)$ is total profit, $P'(x) = R'(x) - C'(x)$ is the **marginal profit**.

Marginal quantities

For any total quantity $Q(x)$ that depends on the number of units x, the **marginal** Q at production level x is $Q'(x)$, with units [dollars, or whatever Q is measured in] per unit. A positive marginal profit at $x = x_0$ means producing more units beyond x_0 increases total profit; a negative marginal profit means producing more units beyond x_0 decreases total profit.

Ví dụ mẫu · Marginal cost, revenue, and profit

A company's cost to produce x office chairs is $C(x) = 0.01x^2 + 20x + 500$ dollars, and the revenue from selling x chairs is $R(x) = 60x - 0.02x^2$ dollars. Find the marginal cost, marginal revenue, and marginal profit at $x = 300$ units, and interpret the marginal profit.

Marginal cost: $C'(x) = 0.02x + 20$, so $C'(300) = 0.02(300) + 20 = 6 + 20 = \$26$ per chair.
Marginal revenue: $R'(x) = 60 - 0.04x$, so $R'(300) = 60 - 0.04(300) = 60 - 12 = \$48$ per chair.
Marginal profit: $P(x) = R(x) - C(x) = -0.03x^2 + 40x - 500$, so $P'(x) = -0.06x + 40$, and $P'(300) = -0.06(300) + 40 = -18 + 40 = \$22$ per chair. (Check: $R'(300) - C'(300) = 48 - 26 = 22$, consistent.)

Interpretation: at a production level of 300 chairs, profit is increasing at approximately $\$22$ for each additional chair produced – since marginal revenue ($\$48$) exceeds marginal cost ($\$26$), producing more chairs beyond 300 is profitable.

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Chi phí biên (marginal cost) $C'(x)$ xấp xỉ chi phí để sản xuất thêm **một** đơn vị kể từ mức x; tương tự cho **doanh thu biên** $R'(x)$ và **lợi nhuận biên** $P'(x) = R'(x) - C'(x)$. Nếu lợi nhuận biên dương tại một mức sản lượng, công ty nên tiếp tục tăng sản lượng (lợi nhuận còn đang tăng); nếu âm, nên giảm sản lượng. Mức sản lượng tối ưu (lợi nhuận cao nhất) chính là nơi $P'(x) = 0$, tức doanh thu biên bằng chi phí biên.

4.3.2 Other applied rates of change

The same idea – differentiate a total-quantity function with respect to its input, then read off sign and units – applies to population counts, volumes, areas, and countless other applied quantities. The essential skill is simply setting up the correct function and units before differentiating. In every case, the input variable does not have to be time: a chemist might track the rate of change of reaction yield with respect to temperature, or a physiologist the rate of change of drug concentration with respect to dosage. Whatever the input variable represents, the same interpretation rules from the beginning of this chapter still govern how the resulting derivative is read: units of output per unit of input, sign gives direction, and magnitude gives speed of change.

Ví dụ mẫu · Rate of change of a population

The number of subscribers to a streaming service, in thousands, t months after launch is modeled for $0 \le t \le 30$ by

$$S(t) = 800 + 120t - 3t^2.$$

Find $S'(t)$, evaluate and interpret $S'(10)$ and $S'(25)$, and determine the month in which the subscriber count stops growing.

$S'(t) = 120 - 6t$ (thousand subscribers per month).
$S'(10) = 120 - 6(10) = 120 - 60 = 60$: at $t = 10$ months, the number of subscribers is *increasing* at a rate of about $60{,}000$ subscribers per month.
$S'(25) = 120 - 6(25) = 120 - 150 = -30$: at $t = 25$ months, the number of subscribers is *decreasing* at a rate of about $30{,}000$ subscribers per month.
Growth stops when $S'(t) = 0$: $120 - 6t = 0 \Rightarrow t = 20$. **At $t = 20$ months**, the subscriber count is momentarily neither increasing nor decreasing (its maximum).

Ví dụ mẫu · Rate of change of a melting volume

A block of ice has volume $V(t) = 1200 - 8t^2$ cubic centimeters, t minutes after being removed from a freezer, for $0 \le t \le 10$. Find the rate at which the volume is changing at $t = 5$ minutes, and interpret the sign and units of the result.

$V'(t) = -16t$, so $V'(5) = -16(5) = -80$ cm^3/min. At $t = 5$ minutes, the volume of the ice block is *decreasing* at a rate of 80 cubic centimeters per minute – the ice is melting.

VN **GIẢI THÍCH TIẾNG VIỆT**

Kỹ thuật chung không đổi dù bối cảnh là gì: xác định hàm biểu diễn **tổng lượng** theo biến độc lập (thường là thời gian), lấy đạo hàm, rồi đọc **dấu** (tăng hay giảm) và **đơn vị** (luôn là đơn vị đại

lượng chia đơn vị biến). Lưu ý sự khác biệt với phần **tốc độ liên hệ (related rates)** ở các mục tiếp theo: ở đây chỉ có *một* hàm phụ thuộc thời gian được cho trực tiếp, còn related rates liên quan *hai (hay nhiều) đại lượng* ràng buộc với nhau bởi một phương trình hình học hoặc đại số.

4.4 Related Rates: The General Method

4.4.1 The general method for related rates

A **related rates** problem involves two or more quantities that are all changing with time and are linked by a fixed geometric or algebraic relationship. Knowing the rate of change of one quantity at a particular instant, the goal is to find the rate of change of another quantity at that same instant.

Every variable in a related rates problem is implicitly a function of time, even though t rarely appears explicitly in the relating equation itself. Differentiating with respect to t is therefore always an application of the **chain rule**: for instance, $\frac{d}{dt}(r^3) = 3r^2\,\frac{dr}{dt}$, because r is secretly $r(t)$, and differentiating $r(t)^3$ with respect to t requires the chain rule exactly as differentiating $\big(g(x)\big)^3$ with respect to x would.

Five-step strategy for related rates

1. **Identify variables.** List every quantity that changes with time, assign it a symbol, and note which rates are given and which rate is unknown. Draw and label a diagram if the problem is geometric.
2. **Write a relating equation.** Find a single equation, valid at *every* instant of time (not just the instant in question), that connects the variables – typically a geometric formula (area, volume, Pythagorean theorem, similar triangles) or a given algebraic relationship.
3. **Differentiate implicitly with respect to t.** Apply $\frac{d}{dt}$ to both sides of the equation, using the chain rule for every variable that depends on t (this is exactly implicit differentiation, treating t as the independent variable).
4. **Substitute known numerical values – *after* differentiating, never before.** Plug in the specific values of the variables and known rates that describe the instant in question.
5. **Solve for the unknown rate**, and state the answer with correct units, including a sentence of interpretation if the problem asks for one.

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Quy trình năm bước cho related rates: (1) liệt kê các đại lượng thay đổi theo thời gian; (2) viết một **phương trình liên hệ** đúng tại **mọi** thời điểm; (3) **đạo hàm ẩn theo** t cả hai vế (dùng chain rule cho mỗi biến phụ thuộc t); (4) **chỉ sau khi đã đạo hàm**, mới thay các giá trị số đã biết vào; (5) giải ra tốc độ cần tìm, kèm đơn vị. Thứ tự bước (3) trước bước (4) là điều quan trọng nhất của toàn bộ chủ đề này.

(!) Lỗi thường gặp: The single most common fatal error in related rates is substituting the known numerical values into the relating equation **before** differentiating. For example, in a problem about a sphere with $V = \frac{4}{3}\pi r^3$, if a student is told $r = 5$ cm at the instant in question and immediately substitutes to get $V = \frac{4}{3}\pi(5)^3 = \frac{500\pi}{3}$ – now a fixed number – and then differentiates, they will get $\frac{dV}{dt} = 0$, since the derivative of a constant is 0. This is wrong. The relating equation $V = \frac{4}{3}\pi r^3$ must be differentiated *first*, while r is still treated as a variable

function of t, giving $\dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$ – *only then* is $r = 5$ substituted into this derivative equation.

Ví dụ mẫu · A first related-rates computation

Suppose x and y are both differentiable functions of t satisfying $x^2 + y^2 = 25$. At the instant when $x = 3$ (and $y = 4 > 0$), $\dfrac{dx}{dt} = 3$. Find $\dfrac{dy}{dt}$ at that instant.

Differentiate implicitly with respect to t:

$$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(25) \implies 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

Now substitute $x = 3$, $y = 4$, $\dfrac{dx}{dt} = 3$ (after differentiating, as required):

$$2(3)(3) + 2(4)\frac{dy}{dt} = 0 \implies 18 + 8\frac{dy}{dt} = 0 \implies \frac{dy}{dt} = -\frac{18}{8} = -\frac{9}{4}.$$

So $\dfrac{dy}{dt} = -2.25$ at that instant.

4.5 Solving Related Rates Problems

4.5.1 An expanding sphere

Ví dụ mẫu · A spherical balloon being inflated

A spherical weather balloon is inflated with helium gas pumped in at a constant rate of 8 m^3/min. Find the rate at which the radius is increasing at the instant the radius is 2 meters.

Step 1 – variables: let V be the volume and r the radius of the balloon, both functions of t. Given: $\dfrac{dV}{dt} = 8$ m^3/min (constant). Find $\dfrac{dr}{dt}$ when $r = 2$ m.

Step 2 – relating equation: $V = \dfrac{4}{3}\pi r^3$.

Step 3 – differentiate with respect to t:

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}.$$

Step 4 – substitute $\dfrac{dV}{dt} = 8$ and $r = 2$:

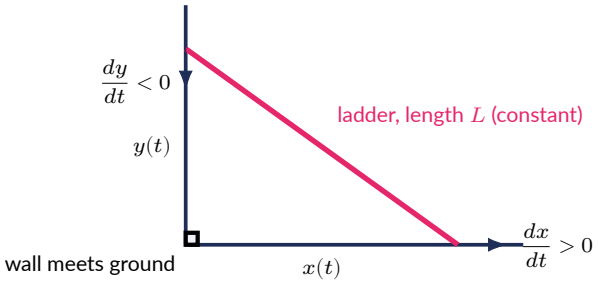
$$8 = 4\pi(2)^2\frac{dr}{dt} = 16\pi\frac{dr}{dt}.$$

Step 5 – solve:

$$\frac{dr}{dt} = \frac{8}{16\pi} = \frac{1}{2\pi} \approx 0.159 \text{ m/min}.$$

At the instant the radius is 2 m, the radius is increasing at about 0.159 meters per minute.

4.5.2 A ladder sliding down a wall



A ladder of fixed length L leans against a vertical wall. As the foot slides away from the wall ($x(t)$ increasing), the top slides down the wall ($y(t)$ decreasing).

Ví dụ mẫu · A ladder sliding down a wall

A 10-meter ladder leans against a vertical wall. The bottom of the ladder is pulled away from the wall at a constant rate of 1.2 m/s. How fast is the top of the ladder sliding down the wall at the instant the bottom of the ladder is 6 m from the wall?

Variables: let $x(t)$ be the distance from the wall to the foot of the ladder, and $y(t)$ the height of the top of the ladder on the wall. Given $\dfrac{dx}{dt} = 1.2$ m/s (constant, since the ladder length is fixed at 10 m). Find $\dfrac{dy}{dt}$ when $x = 6$ m.

Relating equation (Pythagorean theorem, true at every instant): $x^2 + y^2 = 10^2 = 100$.

When $x = 6$: $y^2 = 100 - 36 = 64$, so $y = 8$ m.

Differentiate with respect to t:

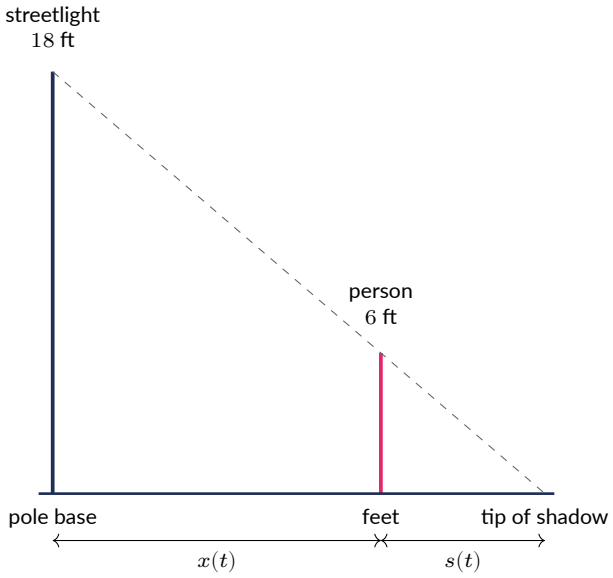
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \implies x\frac{dx}{dt} + y\frac{dy}{dt} = 0.$$

Substitute $x = 6$, $y = 8$, $\dfrac{dx}{dt} = 1.2$:

$$(6)(1.2) + 8\frac{dy}{dt} = 0 \implies 7.2 + 8\frac{dy}{dt} = 0 \implies \frac{dy}{dt} = -\frac{7.2}{8} = -0.9 \text{ m/s}.$$

The top of the ladder is sliding down the wall at 0.9 m/s at the instant the bottom is 6 m from the wall. (The negative sign is expected: y is decreasing.)

4.5.3 A shadow lengthening along the ground



Similar triangles: the large triangle (pole, ground, light ray) and the small triangle (person, ground, same light ray) share the same light ray as hypotenuse, so their height-to-base ratios are equal.

Ví dụ mẫu · A shadow problem via similar triangles

A street lamp is 18 ft above the ground. A person 6 ft tall walks away from the lamp at a constant rate of 5 ft/s. At what rate is the length of the person's shadow increasing? At what rate is the tip of the shadow moving away from the pole?

Variables: let $x(t)$ be the distance of the person from the base of the pole, and $s(t)$ the length of the person's shadow, both in feet. Given $\dfrac{dx}{dt} = 5$ ft/s. Find $\dfrac{ds}{dt}$.

Relating equation: the pole, the person, and the light ray to the tip of the shadow form two similar right triangles (as shown above), so the ratio of height to base is the same for both:

$$\frac{18}{x+s} = \frac{6}{s} \implies 18s = 6(x+s) = 6x + 6s \implies 12s = 6x \implies s = \frac{x}{2}.$$

Because this relationship holds at every instant, it is valid to differentiate directly:

$$\frac{ds}{dt} = \frac{1}{2}\frac{dx}{dt} = \frac{1}{2}(5) = 2.5 \text{ ft/s}.$$

The shadow is lengthening at 2.5 ft/s.

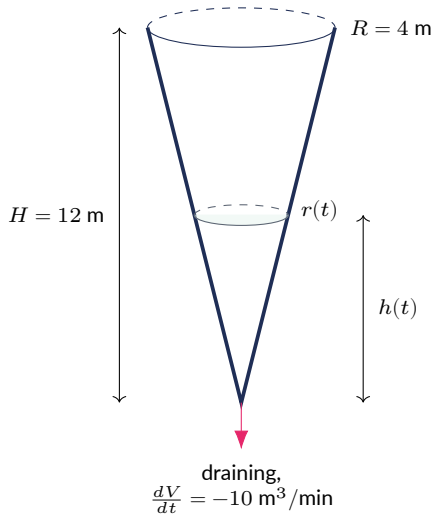
The tip of the shadow is located at distance $x + s$ from the pole, so its speed is

$$\frac{d}{dt}(x+s) = \frac{dx}{dt} + \frac{ds}{dt} = 5 + 2.5 = 7.5 \text{ ft/s}.$$

VN **GIẢI THÍCH TIẾNG VIỆT**

Bài toán "bóng đổ" luôn dựa trên **tam giác đồng dạng**: chiều cao chia đế của tam giác lớn (cột đèn) bằng chiều cao chia đế của tam giác nhỏ (người). Vì tỉ lệ này là một **hằng số hình học** đúng tại mọi thời điểm, ta có thể rút gọn thành một phương trình tuyến tính đơn giản ($s = \frac{x}{2}$ trong ví dụ trên) rồi đạo hàm trực tiếp – không cần dùng công thức đạo hàm của thương số.

4.5.4 A draining conical tank



Inverted conical tank, height $H = 12$ m and top radius $R = 4$ m. At water depth $h(t)$, the water's surface has radius $r(t)$, and similar triangles force $r/h = R/H = 1/3$ at every instant.

Ví dụ mẫu · A draining conical tank

Water drains from an inverted conical tank of height 12 m and top radius 4 m at a constant rate of 10 m^3/min. Find the rate at which the water level is falling at the instant the water is 6 m deep.

Variables: let V be the volume of water, h the depth of water, and r the radius of the water's surface, all functions of t. Given $\dfrac{dV}{dt} = -10$ m^3/min (negative, since draining). Find $\dfrac{dh}{dt}$ when $h = 6$ m.

Relating equation: the volume of a cone is $V = \dfrac{1}{3}\pi r^2 h$. This has *two* time-dependent variables, r and h, so first eliminate r using similar triangles: since the full tank has radius 4 at height 12, at every instant $\dfrac{r}{h} = \dfrac{4}{12} = \dfrac{1}{3}$, i.e. $r = \dfrac{h}{3}$. Substituting,

$$V = \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h = \frac{1}{3}\pi \cdot \frac{h^2}{9} \cdot h = \frac{\pi h^3}{27}.$$

Differentiate with respect to t (only V and h remain, both functions of t):

$$\frac{dV}{dt} = \frac{\pi}{27} \cdot 3h^2 \frac{dh}{dt} = \frac{\pi h^2}{9}\frac{dh}{dt}.$$

Substitute $\dfrac{dV}{dt} = -10$ and $h = 6$:

$$-10 = \frac{\pi(6)^2}{9}\frac{dh}{dt} = \frac{36\pi}{9}\frac{dh}{dt} = 4\pi\frac{dh}{dt}.$$

Solve:

$$\frac{dh}{dt} = \frac{-10}{4\pi} = -\frac{5}{2\pi} \approx -0.796 \text{ m/min}.$$

At the instant the water is 6 m deep, the water level is falling at about 0.796 meters per minute.

(!) Lỗi thường gặp: In the conical tank example, the volume formula $V = \frac{1}{3}\pi r^2 h$ contains *two* variables that both change with time, r and h, but only one rate is wanted (dh/dt) and only one is given (dV/dt). Differentiating $V = \frac{1}{3}\pi r^2 h$ directly (using the product rule on $r^2 h$) would introduce an extra unknown rate dr/dt with no way to find it. The fix is always to first use the fixed geometric relationship between r and h (here, similar triangles, $r = h/3$) to rewrite V as a function of h *alone*, and only then differentiate with respect to t.

4.6 Approximating Values Using Local Linearity

4.6.1 The tangent line as a local approximation

If f is differentiable at $x = a$, the tangent line to the graph of f at the point $(a, f(a))$ is called the **linearization** of f at a:

$$L(x) = f(a) + f'(a)(x - a).$$

Local linearity is the principle that, for x sufficiently close to a, the curve $y = f(x)$ is very well approximated by its tangent line: $f(x) \approx L(x)$. This is the same idea used to *define* the derivative in the first place (the tangent line is the limit of secant lines), used here in reverse – to estimate function values near a point where $f(a)$ and $f'(a)$ are known or easy to compute exactly.

The quality of the approximation $f(x) \approx L(x)$ degrades as x moves farther from a, because the tangent line is a *straight* line while the actual curve may bend away from it. This is why linearization is only reliable "locally" – near the point of tangency.

Linearization is useful precisely because $f(a)$ and $f'(a)$ are often easy to compute exactly (for a "nice" input a, such as a perfect square or a point where a formula simplifies), while $f(x)$ at a nearby but "messy" input x is not. Estimating $f(x)$ by evaluating the much simpler linear function $L(x)$ avoids the need for a calculator or a more complicated exact computation, at the cost of a small, predictable amount of error.

VN **GIẢI THÍCH TIẾNG VIỆT**

Đường tiếp tuyến tại $x = a$, viết dưới dạng $L(x) = f(a) + f'(a)(x - a)$, được gọi là **xấp xỉ tuyến tính (linearization)** của f tại a. Nguyên lý **tuyến tính hoá địa phương (local linearity)**: khi x đủ gần a, đồ thị $y = f(x)$ gần như trùng với đường tiếp tuyến, nên $f(x) \approx L(x)$. Xấp xỉ này càng **kém chính xác** khi x càng **xa** a, vì đường thẳng không thể "bám" theo độ cong của đồ thị mãi mãi.

Ví dụ mẫu · Linearizing the square-root function

Use linearization to approximate $\sqrt{4.1}$, based on $f(x) = \sqrt{x}$ at $a = 4$.

$f(4) = 2$. $f'(x) = \dfrac{1}{2\sqrt{x}}$, so $f'(4) = \dfrac{1}{4}$.

$$L(x) = 2 + \frac{1}{4}(x - 4) \implies L(4.1) = 2 + \frac{1}{4}(0.1) = 2 + 0.025 = 2.025.$$

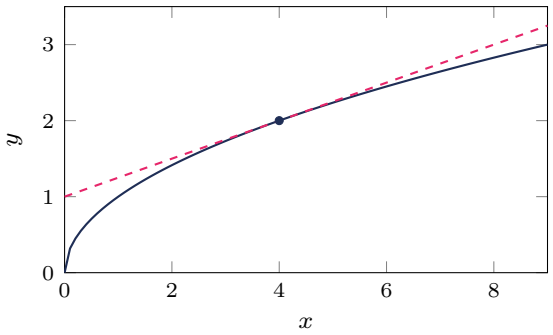
(For comparison, the true value is $\sqrt{4.1} \approx 2.02485$, so the linear approximation is accurate to about 0.00015.)

4.6.2 Overestimates, underestimates, and concavity

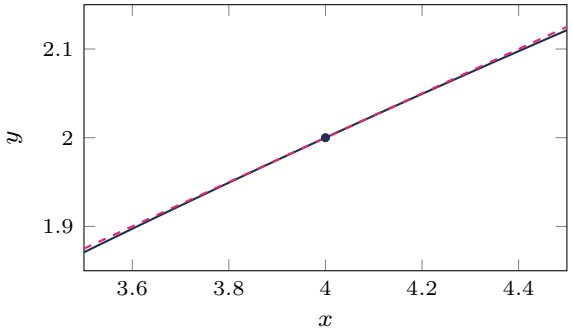
Whether $L(x)$ over- or under-estimates the true value $f(x)$ near $x = a$ is controlled entirely by the **concavity** of f near a – that is, by the sign of f''.

- If $f''(x) > 0$ near a (**concave up**), the tangent line lies *below* the curve, so $L(x)$ **underestimates** $f(x)$.
- If $f''(x) < 0$ near a (**concave down**), the tangent line lies *above* the curve, so $L(x)$ **overestimates** $f(x)$.

For $f(x) = \sqrt{x}$ above, $f''(x) = -\frac{1}{4}x^{-3/2} < 0$ for all $x > 0$, so f is concave down everywhere on its domain; the linearization $L(4.1) = 2.025$ should therefore *overestimate* the true value – and indeed $2.025 > 2.02485$, confirming the concavity argument.



Full view on $[0, 9]$: $f(x) = \sqrt{x}$ (solid) and its tangent line $L(x)$ at $a = 4$ (dashed). The lines visibly separate away from $a = 4$.



Zoomed to $[3.5, 4.5]$: near $a = 4$ the curve and the tangent line are nearly indistinguishable – this is local linearity.

Ví dụ mẫu · Reasoning about over/underestimates without a formula

Suppose $f(3) = 7$, $f'(3) = -2$, and $f''(x) > 0$ for all x near 3. Use linearization to estimate $f(3.2)$, and state whether this estimate is larger or smaller than the true value of $f(3.2)$.

$$L(x) = f(3) + f'(3)(x - 3) = 7 - 2(x - 3) \implies L(3.2) = 7 - 2(0.2) = 7 - 0.4 = 6.6.$$

Since $f''(x) > 0$ near $x = 3$, f is concave up there, so the tangent line lies *below* the curve. Therefore $L(3.2) = 6.6$ is an **underestimate**: the true value of $f(3.2)$ is greater than 6.6.

Ví dụ mẫu · Linearizing the sine function near zero

Use linearization to approximate $\sin(0.1)$ (radians), based on $f(x) = \sin x$ at $a = 0$, and determine whether the approximation is an over- or underestimate.

$f(0) = 0$, $f'(x) = \cos x$, $f'(0) = 1$. So $L(x) = 0 + 1(x - 0) = x$, and $L(0.1) = 0.1$.
Since $f''(x) = -\sin x$, and $\sin x > 0$ for $x \in (0, 0.1]$, we have $f''(x) < 0$ on this interval: f is concave down just to the right of $a = 0$, so the tangent line lies above the curve there, and $L(0.1) = 0.1$ **overestimates** $\sin(0.1)$. (Indeed, $\sin(0.1) \approx 0.09983$, slightly less than 0.1.)

4.7 Differentials

4.7.1 The differential and propagated error

Closely related to linearization is the notion of a **differential**. Writing $\Delta x = dx$ for a small change in x, the **differential** of $y = f(x)$ is defined as

$$dy = f'(x)\,dx.$$

The differential dy approximates the **true change** $\Delta y = f(x+\Delta x) - f(x)$ for small Δx: $\Delta y \approx dy$. This is exactly the same approximation as linearization, restated in differential notation: $f(x+\Delta x) \approx f(x) + f'(x)\,\Delta x$, i.e. $f(x+\Delta x) - f(x) \approx dy$.

VN **GIẢI THÍCH TIẾNG VIỆT**

Vi phân (differential) $dy = f'(x)\,dx$ là một **xấp xỉ** của **độ biến thiên thực** $\Delta y = f(x+\Delta x)-f(x)$, khi $\Delta x = dx$ đủ nhỏ. Về bản chất, đây chính là ý tưởng xấp xỉ tuyến tính viết dưới dạng ký hiệu vi phân – dy luôn là một giá trị **gần đúng**, không phải giá trị chính xác của Δy (trừ khi f là hàm bậc nhất).

Ví dụ mẫu · Comparing the differential to the true change

Let $y = f(x) = x^3$, and let x change from 2 to 2.1 (so $dx = \Delta x = 0.1$). Compute the differential dy and the true change Δy, and compare.

$f'(x) = 3x^2$, so $dy = f'(2)\,dx = 3(2)^2(0.1) = 3(4)(0.1) = 1.2$.
True change: $\Delta y = f(2.1) - f(2) = (2.1)^3 - 2^3 = 9.261 - 8 = 1.261$.
The differential $dy = 1.2$ is close to, but not exactly equal to, the true change $\Delta y = 1.261$ – the difference (0.061) is the error introduced by using the (straight) tangent line instead of the (curved) graph of f.

(!) Lỗi thường gặp: The differential dy is an **approximation** of Δy, not an exact restatement of it – the two are generally different numbers, as the example above shows ($dy = 1.2$ versus $\Delta y = 1.261$). Writing "$dy = \Delta y$" as though they are always identical is incorrect; the correct relationship is $\Delta y \approx dy$ for small dx, with the approximation improving as $dx \to 0$.

Ví dụ mẫu · Propagated error using differentials

The radius of a sphere is measured to be $r = 10$ cm, with a possible measurement error of $dr = \pm 0.1$ cm. Use differentials to estimate the resulting error in the computed volume, and express this as a percent error.

$V = \frac{4}{3}\pi r^3$, so $\frac{dV}{dr} = 4\pi r^2$. At $r = 10$:

$$dV = 4\pi(10)^2\,dr = 400\pi(0.1) = 40\pi \approx 125.66 \text{ cm}^3.$$

The volume computed from the measured radius, $V(10) = \frac{4}{3}\pi(10)^3 = \frac{4000\pi}{3} \approx 4188.79$ cm^3, therefore carries a propagated error of about ± 125.66 cm^3.

Percent error:

$$\frac{dV}{V} \times 100\% = \frac{40\pi}{4000\pi/3} \times 100\% = \frac{40 \times 3}{4000} \times 100\% = 3\%.$$

This matches the general rule for a power relationship $V \propto r^3$: a relative error of $\frac{dr}{r} = \frac{0.1}{10} = 1\%$ in the radius propagates to a relative error of $3 \times 1\% = 3\%$ in the volume (the exponent 3 multiplies the relative error).

4.8 L'Hopital's Rule for Indeterminate Limits

4.8.1 Statement and conditions of the rule

Some limits of the form $\lim\limits_{x\to c}\dfrac{f(x)}{g(x)}$ cannot be evaluated by direct substitution because doing so produces a meaningless expression such as $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ – these are called **indeterminate forms**. L'Hopital's Rule gives a way to evaluate such limits using derivatives.

L'Hopital's Rule. Suppose f and g are differentiable near c (except possibly at c itself), $g'(x)\neq 0$ near c, and

$$\lim_{x\to c}\frac{f(x)}{g(x)} \text{ is of the form } \frac{0}{0} \text{ or } \frac{\pm\infty}{\pm\infty}.$$

Then

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)},$$

provided the limit on the right exists (as a finite number) or is $\pm\infty$. The rule applies equally to one-sided limits and to limits as $x\to\pm\infty$.

Two conditions are essential and frequently overlooked. First, L'Hopital's Rule applies *only* to the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$ (and, after algebraic rewriting, a few related forms) – it must **never** be applied to a limit that is not indeterminate, even if it looks superficially similar. Second, if applying the rule once produces *another* indeterminate form, the rule may be applied again – but the indeterminate form must be verified freshly **before every single application**, not assumed to persist automatically.

It is also worth emphasizing what L'Hopital's Rule is *not*: it does not say to apply the quotient rule to $\dfrac{f(x)}{g(x)}$. Instead, the numerator and denominator are differentiated **separately**, and the new derivatives are placed in a brand-new fraction $\dfrac{f'(x)}{g'(x)}$. Confusing the two procedures is a common source of algebra errors even among students who correctly recognize the indeterminate form.

VN **GIẢI THÍCH TIẾNG VIỆT**

Quy tắc L'Hôpital chỉ áp dụng được cho các **dạng vô định** $\frac{0}{0}$ hoặc $\frac{\infty}{\infty}$: lấy đạo hàm tử số và mẫu số **riêng biệt** (không phải đạo hàm của cả phân thức theo quy tắc thương số!), rồi tính lại giới hạn. Nếu sau một lần áp dụng vẫn còn ra dạng vô định, có thể áp dụng tiếp – nhưng **mỗi lần** đều phải kiểm tra lại xem có đúng là dạng vô định hay không trước khi đạo hàm thêm một lần nữa.

(!) Lỗi thường gặp: L'Hopital's Rule may be applied **only** when the limit is genuinely in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. For example, $\lim\limits_{x\to 0}\dfrac{x+4}{x+2}$ is *not* indeterminate: direct substitution gives $\dfrac{0+4}{0+2}=\dfrac{4}{2}=2$, a perfectly well-defined finite answer. A student who mistakenly applies L'Hopital's Rule anyway – differentiating numerator and denominator to get $\lim\limits_{x\to 0}\dfrac{1}{1}=1$ – obtains the **wrong** answer (1 instead of the correct 2). Before invoking the rule, always substitute first to confirm the limit truly is $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

4.8.2 Worked applications

Ví dụ mẫu · A single application, form zero over zero

Evaluate $\lim_{x\to 0}\dfrac{e^x-1}{x}$.

Direct substitution gives $\dfrac{e^0-1}{0}=\dfrac{0}{0}$, an indeterminate form, so L'Hopital's Rule applies. Differentiate numerator and denominator separately:

$$\lim_{x\to 0}\frac{e^x-1}{x}=\lim_{x\to 0}\frac{e^x}{1}=e^0=1.$$

Check (Taylor series): $e^x=1+x+\frac{x^2}{2}+\cdots$, so $e^x-1=x+\frac{x^2}{2}+\cdots$, and dividing by x gives $1+\frac{x}{2}+\cdots\to 1$ as $x\to 0$. Consistent.

Ví dụ mẫu · A single application, form infinity over infinity

Evaluate $\lim_{x\to\infty}\dfrac{\ln x}{x}$.

As $x\to\infty$, both $\ln x\to\infty$ and $x\to\infty$, so this is $\frac{\infty}{\infty}$. Differentiate numerator and denominator:

$$\lim_{x\to\infty}\frac{\ln x}{x}=\lim_{x\to\infty}\frac{1/x}{1}=\lim_{x\to\infty}\frac{1}{x}=0.$$

This confirms the standard fact that $\ln x$ grows more slowly than x as $x\to\infty$.

Ví dụ mẫu · Two applications, rechecking the indeterminate form each time

Evaluate $\lim_{x\to 0}\dfrac{1-\cos x}{x^2}$.

Check the form: at $x=0$, numerator $=1-\cos 0=0$, denominator $=0$: form $\frac{0}{0}$. Apply L'Hopital once:

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}.$$

Recheck the form before differentiating again: at $x=0$, numerator $=\sin 0=0$, denominator $=2(0)=0$: still $\frac{0}{0}$. Apply L'Hopital a *second* time:

$$\lim_{x\to 0}\frac{\sin x}{2x}=\lim_{x\to 0}\frac{\cos x}{2}=\frac{\cos 0}{2}=\frac{1}{2}.$$

Check (Taylor series): $\cos x=1-\frac{x^2}{2}+\frac{x^4}{24}-\cdots$, so $1-\cos x=\frac{x^2}{2}-\frac{x^4}{24}+\cdots$, and dividing by x^2 gives $\frac{1}{2}-\frac{x^2}{24}+\cdots\to\frac{1}{2}$ as $x\to 0$. Consistent.

Why the indeterminate form must be rechecked before each step

After one application of L'Hopital's Rule, the resulting limit is not automatically indeterminate again. If the new limit already evaluates to a definite finite number or $\pm\infty$ by direct substitution, that *is* the final answer – differentiating a second time would change the expression into something unrelated to the original limit and would generally produce an incorrect result.

4.9 Problem Bank

Bài tập luyện tập

If $C(x)$ is the cost, in dollars, to produce x office chairs, what are the units of $C'(250)$?

(A) chairs
(B) dollars
(C) dollars per chair
(D) chairs per dollar

Lời giải chi tiết

$C'(x) = \dfrac{dC}{dx}$ has units of C (dollars) divided by units of x (chairs): dollars per chair. **(C)**.

Bài tập luyện tập

Let $A(t)$ be the number of gallons of water in a swimming pool t hours after an automatic pump is turned on, and suppose $A(3) = 850$ and $A'(3) = 40$. Write one sentence interpreting $A(3) = 850$, and one sentence interpreting $A'(3) = 40$, including units. Then briefly explain the difference between what each value tells you.

Lời giải chi tiết

$A(3) = 850$: three hours after the pump is turned on, the pool contains 850 gallons of water (a *state*, measured in gallons). $A'(3) = 40$ gallons per hour: at $t = 3$ hours, the amount of water in the pool is *increasing* at a rate of 40 gallons per hour (a *rate*, measured in gallons/hour). $A(3)$ tells you how much water is present at that instant; $A'(3)$ tells you how fast that amount is changing at that instant – the two carry different units and answer different questions.

Bài tập luyện tập

A cup of coffee has temperature $F(t)$, in degrees Fahrenheit, t minutes after being poured, with $F(5) = 145$ and $F'(5) = -6$. A student writes: "The temperature of the coffee at $t = 5$ is -6." Explain what is wrong with this statement, and write a corrected pair of sentences describing $F(5)$ and $F'(5)$.

Lời giải chi tiết

The student has confused the derivative value with the function value: $F'(5) = -6$ is a *rate of change* (units °F per minute), not a temperature, and a temperature cannot correctly be reported as a bare number with no units. Corrected: $F(5) = 145$ means that at $t = 5$ minutes, the coffee's temperature is 145°F. $F'(5) = -6$ °F/min means that at $t = 5$ minutes, the coffee's temperature is *decreasing* at a rate of 6°F per minute.

Bài tập luyện tập

A particle moves along the x-axis with position $s(t) = -t^3 + 3t^2 + 9t$ (meters) for $t \geq 0$. For which interval is the particle moving in the negative direction?

(A) $0 < t < 3$
(B) $t > 3$
(C) $0 < t < 1$
(D) never

Lời giải chi tiết

$v(t) = s'(t) = -3t^2 + 6t + 9 = -3(t^2 - 2t - 3) = -3(t-3)(t+1)$. For $t \geq 0$, the only relevant zero is $t = 3$. Testing $t = 0$: $v(0) = 9 > 0$; testing $t = 4$: $v(4) = -3(16) + 24 + 9 = -15 < 0$. So $v(t) < 0$ for $t > 3$. **(B)**.

Bài tập luyện tập

For the same position function $s(t) = -t^3 + 3t^2 + 9t$, determine whether the particle is speeding up or slowing down at $t = 2$ and at $t = 4$.

Lời giải chi tiết

$v(t) = -3t^2 + 6t + 9$ and $a(t) = v'(t) = -6t + 6 = -6(t-1)$.
At $t = 2$: $v(2) = -3(4) + 12 + 9 = 9 > 0$; $a(2) = -6(2-1) = -6 < 0$. Opposite signs $\Rightarrow$ **slowing down**.
At $t = 4$: $v(4) = -3(16) + 24 + 9 = -15 < 0$; $a(4) = -6(4-1) = -18 < 0$. Same sign $\Rightarrow$ **speeding up**.

Bài tập luyện tập

A particle moves along a line with velocity $v(5) = -8$ m/s. What is the particle's speed at $t = 5$?

(A) -8 m/s
(B) 8 m/s
(C) 0 m/s
(D) cannot be determined

Lời giải chi tiết

Speed is $|v(t)|$, so the speed at $t = 5$ is $|-8| = 8$ m/s. **(B)**.

Bài tập luyện tập

A particle moves along the x-axis with velocity $v(t) = t^2 - 5t + 4$ for $0 \leq t \leq 6$. Find all times in this interval at which the particle changes direction, and justify your answer.

Lời giải chi tiết

Factor: $v(t) = (t-1)(t-4)$, zero at $t = 1, 4$. Testing signs: $v(0) = 4 > 0$; $v(2) = 4 - 10 + 4 = -2 < 0$; $v(5) = 25 - 25 + 4 = 4 > 0$. So $v > 0$ on $[0, 1)$, $v < 0$ on $(1, 4)$, $v > 0$ on $(4, 6]$ – velocity changes sign at both zeros, so the particle **changes direction at** $t = 1$ **and** $t = 4$.

Bài tập luyện tập

The cost to produce x units of a product is $C(x)$ dollars, and $C'(400) = 12.5$. Which statement correctly interprets this value?

(A) It costs exactly \$12.50 to produce the 400th unit.
(B) The average cost per unit at 400 units is \$12.50.
(C) The cost of producing one additional unit beyond 400 is approximately \$12.50.
(D) Total cost is decreasing at 400 units.

Lời giải chi tiết

Marginal cost $C'(x)$ is an instantaneous rate that *approximates* the cost of one more unit – it is neither an exact discrete cost nor an average cost, and it is positive here, so cost is increasing, not decreasing. **(C)**.

Bài tập luyện tập

Using $C(x) = 0.01x^2 + 20x + 500$ and $R(x) = 60x - 0.02x^2$ (dollars) from the worked marginal-analysis example, (a) find the marginal profit at $x = 500$ units and interpret it, and (b) find the production level at which marginal profit equals zero.

Lời giải chi tiết

(a) $C'(x) = 0.02x + 20 \Rightarrow C'(500) = 10 + 20 = \$30$/unit. $R'(x) = 60 - 0.04x \Rightarrow R'(500) = 60 - 20 = \$40$/unit. Marginal profit $P'(500) = R'(500) - C'(500) = 40 - 30 = \$10$/unit: at a production level of 500 units, profit is increasing at about $\$10$ for each additional unit, so increasing production is worthwhile.
(b) $P(x) = R(x) - C(x) = -0.03x^2 + 40x - 500$, so $P'(x) = -0.06x + 40$. Setting $P'(x) = 0$: $x = \dfrac{40}{0.06} = \dfrac{2000}{3} \approx 666.7$ units – the (continuous) profit-maximizing production level, where marginal revenue equals marginal cost.

Bài tập luyện tập

A bacteria population is modeled, in thousands, by $N(t) = 200 + 45t - 1.5t^2$ for $0 \le t \le 20$ hours. (a) Find $N'(t)$ and interpret $N'(6)$, with units. (b) Find the time at which the population is at its maximum, and the maximum population.

Lời giải chi tiết

(a) $N'(t) = 45 - 3t$ (thousand bacteria per hour). $N'(6) = 45 - 18 = 27$: at $t = 6$ hours, the population is increasing at about $27{,}000$ bacteria per hour.
(b) $N'(t) = 0 \Rightarrow 45 - 3t = 0 \Rightarrow t = 15$ hours. $N(15) = 200 + 45(15) - 1.5(15)^2 = 200 + 675 - 337.5 = 537.5$ thousand bacteria, i.e. about $537{,}500$ bacteria.

Bài tập luyện tập

A block of ice has volume $V(t) = 1200 - 8t^2$ cubic centimeters, t minutes after being removed from a freezer, for $0 \le t \le 10$. Find the rate at which the volume is changing at $t = 3$ minutes, and interpret the sign of your answer.

Lời giải chi tiết

$V'(t) = -16t$, so $V'(3) = -48$ cm^3/min. At $t = 3$ minutes, the volume of the ice block is decreasing (melting) at a rate of 48 cubic centimeters per minute.

Bài tập luyện tập

Which of the following is the correct final step in solving a related rates problem?

(A) Substitute all known numerical values into the relating equation, then differentiate.

(B) Differentiate the relating equation implicitly with respect to time, then substitute known numerical values.

(C) Solve for the unknown variable algebraically without ever differentiating.

(D) Substitute known values into the derivative equation before actually differentiating.

Lời giải chi tiết

The relating equation must be differentiated *first*, while all variables are still treated as functions of t; only afterward are the known numerical values substituted into the resulting derivative equation. **(B)**.

Bài tập luyện tập

A circular oil slick spreads on the surface of the ocean so that its radius increases at a constant rate of 1.5 m/min. Find the rate at which the area of the slick is increasing at the instant the radius is 40 m.

Lời giải chi tiết

$A = \pi r^2 \Rightarrow \dfrac{dA}{dt} = 2\pi r \dfrac{dr}{dt}$. At $r = 40$, $\dfrac{dr}{dt} = 1.5$:

$$\frac{dA}{dt} = 2\pi(40)(1.5) = 120\pi \approx 376.99 \text{ m}^2/\text{min}.$$

Bài tập luyện tập

A 10-meter ladder leans against a vertical wall. The bottom of the ladder slides away from the wall at a constant rate of 1.2 m/s. How fast is the top of the ladder sliding down the wall at the instant the bottom is 6 m from the wall?

Lời giải chi tiết

$x^2 + y^2 = 100$. At $x = 6$: $y = \sqrt{100 - 36} = \sqrt{64} = 8$. Differentiate: $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$. Substitute $x = 6$, $y = 8$, $\frac{dx}{dt} = 1.2$:

$$(6)(1.2) + 8\frac{dy}{dt} = 0 \Rightarrow 7.2 + 8\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -0.9 \text{ m/s}.$$

The top of the ladder is sliding down at 0.9 m/s.

Bài tập luyện tập

A street lamp is 20 ft above the ground. A person 5 ft tall walks away from the lamp at a constant rate of 6 ft/s. At what rate is the length of the person's shadow increasing?

Lời giải chi tiết

By similar triangles, $\dfrac{20}{x+s} = \dfrac{5}{s} \Rightarrow 20s = 5(x+s) = 5x + 5s \Rightarrow 15s = 5x \Rightarrow s = \dfrac{x}{3}$.

Differentiating, $\frac{ds}{dt} = \frac{1}{3}\frac{dx}{dt} = \frac{1}{3}(6) = 2$ ft/s. (The tip of the shadow moves at $\frac{dx}{dt} + \frac{ds}{dt} = 6 + 2 = 8$ ft/s.)

Bài tập luyện tập

An inverted conical tank has height 9 m and top radius 3 m. Water is pumped into the tank at a rate of 2 m^3/min. Find the rate at which the water level is rising at the instant the water is 4 m deep.

Lời giải chi tiết

By similar triangles, $r/h = 3/9 = 1/3$, so $r = h/3$, and $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h = \frac{\pi h^3}{27}$. Then $\frac{dV}{dt} = \frac{\pi h^2}{9}\frac{dh}{dt}$. At $h = 4$, $\frac{dV}{dt} = 2$:

$$2 = \frac{\pi(16)}{9}\frac{dh}{dt} = \frac{16\pi}{9}\frac{dh}{dt} \implies \frac{dh}{dt} = \frac{18}{16\pi} = \frac{9}{8\pi} \approx 0.358 \text{ m/min}.$$

Bài tập luyện tập

Car A leaves an intersection traveling east at a constant 50 mph. At the same moment, Car B leaves the same intersection traveling north at a constant 30 mph. Find the rate at which the distance between the two cars is increasing at the instant Car A is 8 miles east of the intersection and Car B is 6 miles north of it.

Lời giải chi tiết

Let x be Car A's distance east of the intersection and y Car B's distance north of it, with $\frac{dx}{dt} = 50$ mph and $\frac{dy}{dt} = 30$ mph. The distance between the cars is $z = \sqrt{x^2 + y^2}$, so $z^2 = x^2 + y^2$. Differentiating with respect to t: $2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$, i.e. $z\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$.
At $x = 8$, $y = 6$: $z = \sqrt{8^2 + 6^2} = \sqrt{100} = 10$ miles. Substituting:

$$10\frac{dz}{dt} = (8)(50) + (6)(30) = 400 + 180 = 580 \implies \frac{dz}{dt} = \frac{580}{10} = 58 \text{ mph}.$$

Bài tập luyện tập

A rectangle always has length equal to twice its width, $L = 2w$, where both dimensions change with time. Which equation correctly relates $\frac{dA}{dt}$ to $\frac{dw}{dt}$?

(A) $\frac{dA}{dt} = 2w\frac{dw}{dt}$

(B) $\frac{dA}{dt} = 4w\frac{dw}{dt}$

(C) $\frac{dA}{dt} = w^2\frac{dw}{dt}$

(D) $\frac{dA}{dt} = 2\frac{dw}{dt}$

Lời giải chi tiết

$A = Lw = (2w)w = 2w^2$, so $\frac{dA}{dt} = 4w\frac{dw}{dt}$. **(B).**

Bài tập luyện tập

Each side of a square is increasing at a rate of 3 cm/s. Find the rate at which the area of the square is increasing at the instant the side length is 10 cm.

Lời giải chi tiết

$A = s^2 \Rightarrow \dfrac{dA}{dt} = 2s\dfrac{ds}{dt} = 2(10)(3) = 60$ cm^2/s.

Bài tập luyện tập

If f is differentiable at $x = a$, the linearization of f at a is given by:

(A) $L(x) = f(a) + f'(x)(x - a)$
(B) $L(x) = f(x) + f'(a)(x - a)$
(C) $L(x) = f(a) + f'(a)(x - a)$
(D) $L(x) = f'(a) + f(a)(x - a)$

Lời giải chi tiết

The linearization is the tangent line at a: $L(x) = f(a) + f'(a)(x - a)$, using the constants $f(a)$ and $f'(a)$, not $f(x)$ or $f'(x)$. **(C)**.

Bài tập luyện tập

Let $f(x) = \ln x$. (a) Find the linearization $L(x)$ of f at $a = 1$. (b) Use $L(x)$ to approximate $\ln(1.05)$. (c) Is this approximation an overestimate or underestimate of the true value? Justify using the concavity of f.

Lời giải chi tiết

(a) $f(1) = 0$, $f'(x) = 1/x \Rightarrow f'(1) = 1$, so $L(x) = 0 + 1(x - 1) = x - 1$.
(b) $L(1.05) = 1.05 - 1 = 0.05$.
(c) $f''(x) = -1/x^2 < 0$ for all $x > 0$, so f is concave down: the tangent line lies above the curve, so $L(1.05) = 0.05$ is an **overestimate**. (True value $\ln(1.05) \approx 0.04879$, confirming $0.05 > 0.04879$.)

Bài tập luyện tập

Use linearization based on $f(x) = x^{1/3}$ at $a = 27$ to approximate $\sqrt[3]{28}$, and state whether the approximation over- or underestimates the true value.

Lời giải chi tiết

$f(27) = 3$, $f'(x) = \frac{1}{3}x^{-2/3} \Rightarrow f'(27) = \frac{1}{3} \cdot \frac{1}{9} = \frac{1}{27}$. So $L(x) = 3 + \frac{1}{27}(x - 27)$, and

$$L(28) = 3 + \frac{1}{27}(1) = 3 + 0.0370 \approx 3.0370.$$

Since $f''(x) = -\frac{2}{9}x^{-5/3} < 0$ for $x > 0$, f is concave down, so $L(28)$ **overestimates** $\sqrt[3]{28}$ (true value ≈ 3.0366).

Bài tập luyện tập

The differential dy of a differentiable function $y = f(x)$ is defined as:

(A) $dy = f(x + dx) - f(x)$

(B) $dy = f'(x)\,dx$

(C) $dy = \dfrac{f(x + dx) - f(x)}{dx}$

(D) $dy = f''(x)\,dx$

Lời giải chi tiết

By definition, $dy = f'(x)\,dx$ – an approximation of the true change $\Delta y = f(x + dx) - f(x)$. **(B)**.

Bài tập luyện tập

The radius of a circular disk is measured to be 8 cm, with a possible measurement error of ± 0.05 cm. Use differentials to estimate the propagated error in the calculated area of the disk, and find the approximate percent error.

Lời giải chi tiết

$A = \pi r^2 \Rightarrow dA = 2\pi r\,dr$. At $r = 8$, $dr = 0.05$: $dA = 2\pi(8)(0.05) = 0.8\pi \approx 2.51$ cm^2. $A(8) = \pi(64) = 64\pi$. Percent error $= \dfrac{dA}{A} \times 100\% = \dfrac{0.8\pi}{64\pi} \times 100\% = 1.25\%$. (Consistent with the shortcut: relative error in A is twice the relative error in r, since $A \propto r^2$: $2 \times \frac{0.05}{8} = 2 \times 0.625\% = 1.25\%$.)

Bài tập luyện tập

Which of the following limits is in an indeterminate form to which L'Hopital's Rule can be directly applied?

(A) $\lim\limits_{x \to 0} \dfrac{x + 3}{x + 5}$

(B) $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$

(C) $\lim\limits_{x \to 2} \dfrac{x - 2}{x^2}$

(D) $\lim\limits_{x \to \infty} \dfrac{1}{x}$

Lời giải chi tiết

(A) evaluates directly to $3/5$; (C) evaluates directly to $0/4 = 0$; (D) evaluates directly to 0 – none of these are indeterminate. Only (B) gives $\sin(0)/0 = 0/0$, a genuine indeterminate form. **(B)**.

Bài tập luyện tập

Evaluate $\lim\limits_{x \to 0} \dfrac{\tan x}{x}$.

Lời giải chi tiết

Form $\frac{0}{0}$. L'Hopital: $\lim\limits_{x \to 0} \dfrac{\sec^2 x}{1} = \sec^2(0) = \dfrac{1}{\cos^2(0)} = 1$.

Bài tập luyện tập

Evaluate $\lim_{x\to\infty} \dfrac{5x+7}{2x-3}$ using L'Hopital's Rule, and verify your answer using an algebraic method.

Lời giải chi tiết

Form $\frac{\infty}{\infty}$. L'Hopital: $\lim_{x\to\infty} \frac{5}{2} = \frac{5}{2}$. **Verify algebraically:** divide numerator and denominator by x: $\dfrac{5+7/x}{2-3/x} \to \dfrac{5}{2}$ as $x \to \infty$. Consistent.

Bài tập luyện tập

Evaluate $\lim_{x\to 0} \dfrac{x-\sin x}{x^3}$.

Lời giải chi tiết

At $x = 0$: form $\frac{0}{0}$. Apply L'Hopital: $\lim_{x\to 0} \dfrac{1-\cos x}{3x^2}$ – recheck: still $\frac{0}{0}$ at $x = 0$. Apply again: $\lim_{x\to 0} \dfrac{\sin x}{6x}$ – recheck: still $\frac{0}{0}$. Apply a third time: $\lim_{x\to 0} \dfrac{\cos x}{6} = \dfrac{\cos 0}{6} = \dfrac{1}{6}$. (Consistent with the Taylor series $\sin x = x - \frac{x^3}{6} + \cdots$, giving $x - \sin x \approx \frac{x^3}{6}$, so the ratio $\to \frac{1}{6}$.)

Bài tập luyện tập

A student wants to evaluate $\lim_{x\to 0} \dfrac{x+4}{x+2}$ and applies L'Hopital's Rule to obtain $\lim_{x\to 0} \dfrac{1}{1} = 1$. Explain the error in this reasoning, and give the correct value of the limit.

Lời giải chi tiết

Direct substitution gives $\dfrac{0+4}{0+2} = \dfrac{4}{2} = 2$, a well-defined finite value – this limit is **not** an indeterminate form, so L'Hopital's Rule does not apply at all. Differentiating numerator and denominator anyway produces an unrelated (and incorrect) result. The correct value of the limit is **2**, obtained by direct substitution.

Bài tập luyện tập

A particle's velocity $v(t)$, in m/s, is recorded at selected times:

t (s)	0	2	4	6	8
$v(t)$ (m/s)	5	2	-3	-1	4

Assuming v is continuous, between which consecutive recorded times can you guarantee (by the Intermediate Value Theorem) that the particle changed direction? Justify your answer.

Lời giải chi tiết

A direction change requires v to cross zero, i.e. change sign. Reading the signs: $+, +, -, -, +$ at $t = 0, 2, 4, 6, 8$. The sign changes from $+$ (at $t = 2$) to $-$ (at $t = 4$), so by the Intermediate

Value Theorem v must equal 0 somewhere in $(2, 4)$ – a guaranteed direction change on $(2, 4)$. The sign changes again from $-$ (at $t = 6$) to $+$ (at $t = 8$), guaranteeing a second direction change on $(6, 8)$. No sign change occurs between $t = 0$ and $t = 2$, nor between $t = 4$ and $t = 6$, so no direction change is guaranteed on those intervals.

Analytical Applications of Differentiation

Mục tiêu Unit: Định lý Giá trị Cực trị (Extreme Value Theorem) và điểm tới hạn (critical points); Định lý Rolle và Định lý Giá trị Trung bình (Mean Value Theorem); xét tính đơn điệu (đồng biến/nghịch biến) bằng dấu của đạo hàm bậc nhất và cách lập bảng xét dấu; Quy tắc Đạo hàm Bậc nhất (First Derivative Test) để phân loại cực trị địa phương; Quy tắc Điểm Ứng viên (Candidates Test) để tìm cực trị tuyệt đối trên một đoạn đóng; tính lồi–lõm (concavity) và Quy tắc Đạo hàm Bậc hai (Second Derivative Test); điểm uốn (inflection points); phác họa đồ thị hàm số từ thông tin về f' và f'', và mối liên hệ ngược lại giữa đồ thị của f, f', f''; giải các bài toán tối ưu hóa (optimization) có ràng buộc; và ứng dụng đạo hàm để phân tích các quan hệ cho dưới dạng ẩn (implicit relations).

5.1 Extreme Values and Critical Points

Many of the most useful applications of the derivative — from finding the cheapest way to manufacture a container to locating the highest point on a roller-coaster track — boil down to a single question: where does a function attain its largest or smallest value? This chapter develops a complete toolkit, built entirely on the sign of f' and f'', for answering that question rigorously.

5.1.1 The Extreme Value Theorem

A function f has an **absolute (global) maximum** at $x = c$ on an interval I if $f(c) \geq f(x)$ for every x in I; it has an **absolute minimum** at $x = c$ if $f(c) \leq f(x)$ for every x in I. Not every function attains such extreme values — for instance, $f(x) = x$ on the open interval $(0, 1)$ gets arbitrarily close to 0 and to 1 but never actually reaches either value. The following theorem gives a simple, powerful guarantee.

Extreme Value Theorem (EVT). If f is **continuous** on a **closed, bounded interval** $[a, b]$, then f attains an absolute maximum value and an absolute minimum value on $[a, b]$ — each at least once, at some point(s) in $[a, b]$.

Both hypotheses matter. If the interval is not closed (e.g., $(0, 1)$ or $[0, 1)$), or if f has even a single discontinuity inside $[a, b]$ (for instance a vertical asymptote or a jump), the guarantee can fail: the function may increase or decrease without bound, or approach a value without ever attaining it. The EVT does not tell us *where* the extrema occur — only that they exist — so we still need a systematic search procedure, which is exactly what the rest of this chapter builds.

VN **GIẢI THÍCH TIẾNG VIỆT**

Định lý Giá trị Cực trị (EVT) khẳng định: nếu f **liên tục** trên một đoạn **đóng và bị chặn** $[a, b]$, thì f chắc chắn đạt được giá trị lớn nhất (max) và giá trị nhỏ nhất (min) tuyệt đối trên đoạn đó. Cả hai điều kiện — liên tục và đoạn đóng — đều bắt buộc: nếu khoảng không đóng (ví dụ $(0, 1)$) hoặc hàm số có điểm gián đoạn bên trong đoạn, định lý có thể không còn đúng nữa. EVT chỉ đảm bảo sự **tồn tại** của cực trị, chứ không nói cực trị nằm ở đâu — ta cần thêm công cụ (đạo hàm) để xác định vị trí cụ thể.

5.1.2 Critical Points

The search for extrema begins by locating a short list of ”candidate” locations.

Critical point

A number c is a **critical point** of f if c is in the domain of f, and either

$$f'(c) = 0 \qquad \text{or} \qquad f'(c) \text{ is undefined.}$$

Geometrically, this means the graph of f has a horizontal tangent line at $x = c$ (when $f'(c) = 0$), or a corner, cusp, or vertical tangent at $x = c$ (when $f'(c)$ fails to exist).

Critical points matter because of a basic fact: if f has a local maximum or local minimum at an interior point c of its domain, then c *must* be a critical point. (If $f'(c)$ existed and were nonzero, the function would be strictly increasing or decreasing right through c, ruling out a local extremum there.) This gives a necessary — but, as we will see repeatedly in this chapter, *not sufficient* — condition: every local extremum is a critical point, but not every critical point is a local extremum.

VN **GIẢI THÍCH TIẾNG VIỆT**

Điểm tới hạn (critical point) là điểm c thuộc tập xác định của f mà tại đó $f'(c) = 0$ **hoặc** $f'(c)$ không xác định (ví dụ tại góc nhọn, điểm gấp khúc, hoặc tiếp tuyến đứng). Mọi cực trị địa phương (local max/min) đều phải là điểm tới hạn — đây là điều kiện **cần**. Nhưng điều ngược lại không đúng: một điểm tới hạn **chưa chắc** là cực trị (điều kiện không phải là **đủ**) — ta sẽ thấy rất nhiều ví dụ về việc này trong chương này.

Ví dụ mẫu · Locating critical points

Find all critical points of $f(x) = x^3 - 3x^2 - 9x + 5$. Then find the critical point of $h(x) = (x-1)^{2/3} + 2$.

Part 1. Differentiate: $f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x-3)(x+1)$. Since f' is a polynomial, it is defined everywhere, so critical points occur only where $f'(x) = 0$:

$$3(x-3)(x+1) = 0 \implies x = -1 \text{ or } x = 3.$$

Both values are in the domain of f (all real numbers), so f has two critical points: $x = -1$ and $x = 3$. This cubic will serve as a running example through the next several sections.

Part 2. Differentiate using the chain rule: $h'(x) = \frac{2}{3}(x-1)^{-1/3} = \frac{2}{3\sqrt[3]{x-1}}$. This expression is undefined at $x = 1$ (division by zero), but $h(1) = 0^{2/3} + 2 = 2$ is defined, so $x = 1$ *is* in the domain of h. Therefore $x = 1$ is a critical point of h — one where the derivative fails to exist (the graph has a sharp cusp there, since h behaves like $|x-1|^{2/3}$ near $x = 1$).

The phrase "c is in the domain of f" in the definition is not a technicality to skip over — it actively rules out points where f itself is undefined, even if the algebraic form of f' appears to vanish or blow up there. For instance, $k(x) = \dfrac{x^2 - 1}{x - 1}$ simplifies to $x + 1$ algebraically, but $x = 1$ is **excluded** from the domain of k (division by zero in the original expression); consequently $x = 1$ can never be a critical point of k, no matter what the simplified derivative suggests. Likewise, $m(x) = \dfrac{1}{x}$ has $m'(x) = -\dfrac{1}{x^2}$, which is never zero and is undefined only at $x = 0$ — but $x = 0$ is not in the domain of m either, so m has **no critical points at all**. Always check domain membership *before* declaring a critical point.

5.2 Rolle's Theorem and the Mean Value Theorem

5.2.1 Rolle's Theorem

Rolle's Theorem is the simplest case of a family of results connecting the *average* behavior of a function on an interval to its *instantaneous* behavior at a single point inside that interval.

Rolle's Theorem. If f is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$, then there exists at least one number c in (a, b) such that

$$f'(c) = 0.$$

Geometrically: if a smooth curve starts and ends at the same height, it must have at least one point in between where the tangent line is horizontal — the curve cannot climb away from its starting height and come back down (or vice versa) without leveling off somewhere.

VN **GIẢI THÍCH TIẾNG VIỆT**

Định lý Rolle: nếu f liên tục trên $[a, b]$, khả vi trên (a, b), và $f(a) = f(b)$ (hai đầu mút có cùng độ cao), thì tồn tại ít nhất một điểm c bên trong khoảng (a, b) sao cho $f'(c) = 0$ — tức tiếp tuyến tại c nằm ngang. Về mặt hình học: đường cong xuất phát và kết thúc ở cùng một độ cao thì chắc chắn phải "làm phẳng" (tiếp tuyến ngang) ít nhất một lần ở giữa.

Ví dụ mẫu · Applying Rolle's Theorem

Verify that Rolle's Theorem applies to $f(x) = x^3 - 3x^2$ on $[0, 3]$, and find the guaranteed value of c.

Check hypotheses: f is a polynomial, so it is continuous on $[0, 3]$ and differentiable on $(0, 3)$. Also $f(0) = 0 - 0 = 0$ and $f(3) = 27 - 27 = 0$, so $f(0) = f(3)$. All three hypotheses hold, so Rolle's Theorem guarantees at least one $c \in (0, 3)$ with $f'(c) = 0$.

Find c: $f'(x) = 3x^2 - 6x = 3x(x - 2)$. Setting $f'(x) = 0$ gives $x = 0$ or $x = 2$. The value $x = 0$ is an *endpoint*, not inside the open interval $(0, 3)$, so it does not count. The guaranteed value is

$$c = 2.$$

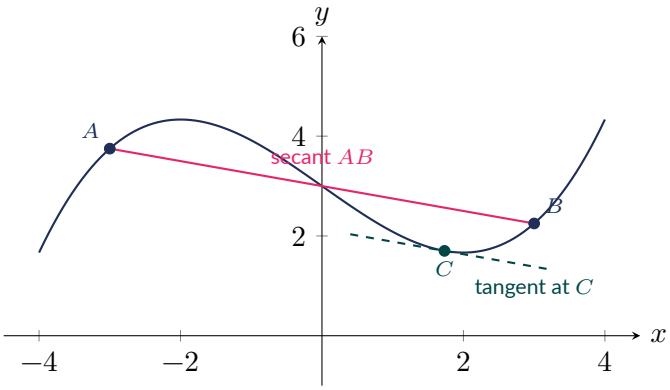
5.2.2 The Mean Value Theorem

Rolle's Theorem is the special case of a more general statement in which the two endpoint heights need not be equal.

Mean Value Theorem (MVT). If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists at least one number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The right-hand side is the slope of the **secant line** joining the endpoints $(a, f(a))$ and $(b, f(b))$ — the *average rate of change* of f over $[a, b]$. The MVT says this average rate of change is always achieved as an actual **instantaneous** rate of change at some interior point: the tangent line at $x = c$ is **parallel** to the secant line. (Rolle's Theorem is the special case where $f(a) = f(b)$, so the secant is horizontal and $f'(c) = 0$.)



Geometric meaning of the MVT: for $f(x) = \frac{1}{12}x^3 - x + 3$ on $[-3, 3]$, the secant line through $A(-3, 3.75)$ and $B(3, 2.25)$ has slope $-\frac{1}{4}$; the tangent line at $C \approx (1.73, 1.70)$ has the same slope, so it is parallel to the secant.

VN **GIẢI THÍCH TIẾNG VIỆT**

Định lý Giá trị Trung bình (MVT): nếu f liên tục trên $[a, b]$ và khả vi trên (a, b), luôn tồn tại một điểm c bên trong sao cho hệ số góc tiếp tuyến tại c **bằng đúng** hệ số góc của đường thẳng nối hai đầu mút (cát tuyến, secant line) — nói cách khác, "tốc độ trung bình" trên cả đoạn luôn được đạt tới như một "tốc độ tức thời" tại ít nhất một thời điểm nào đó bên trong. Định lý Rolle chỉ là trường hợp riêng khi $f(a) = f(b)$ (cát tuyến nằm ngang nên $f'(c) = 0$).

Ví dụ mẫu · Finding the guaranteed c

Let $f(x) = x^3 - 2x + 1$ on $[-1, 2]$. Verify the hypotheses of the MVT and find every value of c guaranteed by the theorem.

Hypotheses: f is a polynomial, so it is continuous on $[-1, 2]$ and differentiable on $(-1, 2)$. The MVT applies.

Average rate of change: $f(-1) = -1 + 2 + 1 = 2$ and $f(2) = 8 - 4 + 1 = 5$, so

$$\frac{f(2) - f(-1)}{2 - (-1)} = \frac{5 - 2}{3} = 1.$$

Solve $f'(c) = 1$: $f'(x) = 3x^2 - 2$, so

$$3c^2 - 2 = 1 \implies 3c^2 = 3 \implies c^2 = 1 \implies c = \pm 1.$$

The value $c = -1$ is the left endpoint, not an interior point, so it is rejected. The guaranteed value is

$$c = 1,$$

which indeed lies in $(-1, 2)$.

(!) Lỗi thường gặp: Đừng quên kiểm tra **cả hai** giả thiết của MVT/Rolle trước khi áp dụng: liên tục trên đoạn đóng $[a, b]$ **và** khả vi trên khoảng mở (a, b). Một hàm có góc nhọn bên trong đoạn (ví dụ $f(x) = |x - 1|$ trên $[-1, 3]$) là liên tục nhưng **không khả vi** tại $x = 1$ — MVT **không** áp dụng được, dù hàm trông "liền mạch". Ngoài ra, khi giải phương trình $f'(c) = \ldots$ ra nhiều nghiệm, phải loại bỏ nghiệm nằm **ngoài** khoảng mở (a, b) (kể cả khi nghiệm đó trùng với đầu mút).

5.3 Increasing and Decreasing Functions

5.3.1 The Sign of the Derivative

The single most important consequence of the Mean Value Theorem is a rigorous link between the *sign* of f' and the *monotonic behavior* of f.

Monotonicity from f'

Let f be differentiable on an open interval I.

- If $f'(x) > 0$ for every x in I, then f is **increasing** on I.
- If $f'(x) < 0$ for every x in I, then f is **decreasing** on I.

(This follows directly from the MVT: for any $x_1 < x_2$ in I, $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$ for some c between them, and the sign of $f'(c)$ determines the sign of $f(x_2) - f(x_1)$.)

VN **GIẢI THÍCH TIẾNG VIỆT**

Đây là ứng dụng quan trọng nhất của MVT: dấu của f' trên một khoảng quyết định hoàn toàn tính đơn điệu của f trên khoảng đó. $f'(x) > 0$ trên cả khoảng $\Rightarrow$ f **đồng biến** (tăng); $f'(x) < 0$ trên cả khoảng $\Rightarrow$ f **nghịch biến** (giảm). Đây chính là công cụ ta sẽ dùng liên tục trong suốt chương này để phân tích hình dạng đồ thị.

5.3.2 Constructing and Reading Sign Charts

Because f' can only change sign at a critical point (or at a point of discontinuity of f'), the critical points found in Section 5.1 split the domain into open intervals on which f' has **constant sign**. A **sign chart** (or sign diagram) organizes this: mark the critical points on a number line, then test one convenient point from each resulting interval in f' to determine the sign on that whole interval.

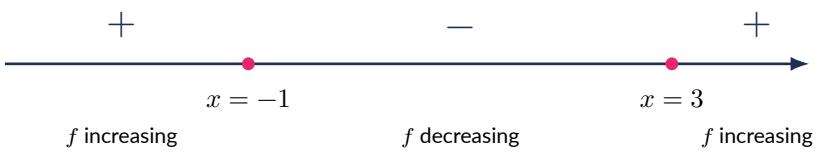
Ví dụ mẫu · Building a sign chart

Determine the intervals on which $f(x) = x^3 - 3x^2 - 9x + 5$ (from the earlier example, with critical points $x = -1$ and $x = 3$) is increasing and decreasing.

Recall $f'(x) = 3(x - 3)(x + 1)$. The critical points $x = -1$ and $x = 3$ divide the real line into three intervals: $(-\infty, -1)$, $(-1, 3)$, $(3, \infty)$. Test one point in each:

- $x = -2$: $f'(-2) = 3(-2 - 3)(-2 + 1) = 3(-5)(-1) = 15 > 0$.
- $x = 0$: $f'(0) = 3(0 - 3)(0 + 1) = 3(-3)(1) = -9 < 0$.
- $x = 4$: $f'(4) = 3(4 - 3)(4 + 1) = 3(1)(5) = 15 > 0$.

So f is **increasing** on $(-\infty, -1)$, **decreasing** on $(-1, 3)$, and **increasing** on $(3, \infty)$.



Sign chart for $f'(x) = 3(x - 3)(x + 1)$: f increases on $(-\infty, -1)$, decreases on $(-1, 3)$, increases on $(3, \infty)$. The sign switches from $+$ to $-$ at $x = -1$ and from $-$ to $+$ at $x = 3$.

5.4 The First Derivative Test

The sign chart from the previous example does more than describe monotonic behavior — it tells us exactly which critical points are local extrema, and of which type.

First Derivative Test. Suppose c is a critical point of a continuous function f.

- If f' changes from **positive to negative** at c (increasing, then decreasing), f has a **local maximum** at c.
- If f' changes from **negative to positive** at c (decreasing, then increasing), f has a **local minimum** at c.
- If f' does **not change sign** at c (same sign on both sides), f has **no local extremum** at c — the graph merely levels off (or has a cusp) momentarily.

Applying this to the running example: at $x = -1$, f' changes from $+$ to $-$, so $f(x) = x^3 - 3x^2 - 9x + 5$ has a **local maximum** at $x = -1$, with value $f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 5 = -1 - 3 + 9 + 5 = 10$. At $x = 3$, f' changes from $-$ to $+$, so f has a **local minimum** at $x = 3$, with value $f(3) = 27 - 27 - 27 + 5 = -22$. Every critical point of this particular function turns out to be an extremum — but that is not guaranteed in general, as the next example and the pitfall below make clear.

VN **GIẢI THÍCH TIẾNG VIỆT**

Quy tắc Đạo hàm Bậc nhất (First Derivative Test): nhìn vào **sự đổi dấu** của f' khi đi qua điểm tới hạn c. Đổi từ $+$ sang $-$ $\Rightarrow$ **cực đại địa phương**. Đổi từ $-$ sang $+$ $\Rightarrow$ **cực tiểu địa phương**. Nếu **không đổi dấu** (cùng dấu ở hai bên) $\Rightarrow$ **không phải cực trị** — đây là trường hợp học sinh hay bỏ sót.

Ví dụ mẫu · First Derivative Test with xe^{-x}

Find and classify all local extrema of $f(x) = xe^{-x}$.

Derivative (product rule): $f'(x) = (1)e^{-x} + x(-e^{-x}) = e^{-x}(1 - x)$. Since $e^{-x} > 0$ for every real x, the only critical point comes from $1 - x = 0$, i.e. $x = 1$ (and f' is defined for all x, so there are no other critical points).

Sign chart: the sign of $f'(x) = e^{-x}(1 - x)$ matches the sign of $(1 - x)$, since $e^{-x} > 0$ always.

- For $x < 1$: $1 - x > 0 \Rightarrow f'(x) > 0$ (increasing).
- For $x > 1$: $1 - x < 0 \Rightarrow f'(x) < 0$ (decreasing).

f' changes from $+$ to $-$ at $x = 1$, so f has a **local maximum** at $x = 1$, with value $f(1) = 1 \cdot e^{-1} = \dfrac{1}{e} \approx 0.368$.

(!) Lỗi thường gặp: "Điểm tới hạn" **không đồng nghĩa** với "cực trị". Xét $g(x) = (x - 2)^3 + 1$: $g'(x) = 3(x - 2)^2 \geq 0$ với mọi x, và $g'(2) = 0$ nên $x = 2$ **là** điểm tới hạn. Nhưng $g'(x) > 0$ ở **cả hai phía** của $x = 2$ (không đổi dấu) — hàm số vẫn đồng biến xuyên suốt, chỉ "khựng lại" tức thời tại $x = 2$. Vậy $x = 2$ **không phải** là cực trị. Luôn kiểm tra sự đổi dấu, đừng vội kết luận chỉ vì $f'(c) = 0$.

5.5 The Candidates Test for Absolute Extrema

The First Derivative Test locates and classifies *local* extrema. When a function is continuous on a **closed interval** $[a, b]$, the EVT guarantees that *absolute* extrema exist as well — and locating them is even simpler than running a full sign chart.

Candidates Test. If f is continuous on $[a, b]$, the absolute maximum and absolute minimum of f on $[a, b]$ each occur either at a critical point inside (a, b), or at one of the two endpoints a or b. To find the absolute extrema:

1. Find all critical points of f that lie inside (a, b).
2. Evaluate f at every such critical point *and* at both endpoints a, b.
3. The largest value found is the absolute maximum; the smallest is the absolute minimum.

VN **GIẢI THÍCH TIẾNG VIỆT**

Quy tắc Điểm Ứng viên (Candidates Test) là cách nhanh nhất để tìm cực trị **tuyệt đối** trên một đoạn đóng $[a, b]$: chỉ cần tính giá trị f tại **tất cả điểm tới hạn nằm trong đoạn** và tại **cả hai đầu mút**, rồi so sánh. Giá trị lớn nhất trong danh sách là max tuyệt đối, giá trị nhỏ nhất là min tuyệt đối — không cần lập bảng xét dấu đầy đủ như First Derivative Test.

Ví dụ mẫu · Candidates Test

Find the absolute maximum and absolute minimum of $f(x) = x^3 - 3x^2 - 9x + 5$ (the running example) on the closed interval $[-2, 4]$.

Critical points inside $(-2, 4)$: from before, $f'(x) = 3(x - 3)(x + 1) = 0$ at $x = -1$ and $x = 3$; both lie inside $(-2, 4)$.

Evaluate f at critical points and endpoints:

$$f(-2) = -8 - 12 + 18 + 5 = 3, \quad f(-1) = 10, \quad f(3) = -22, \quad f(4) = 64 - 48 - 36 + 5 = -15.$$

x	-2 (endpoint)	-1 (critical pt.)	3 (critical pt.)	4 (endpoint)
$f(x)$	3	10	-22	-15

Comparing all four values, the largest is 10 and the smallest is -22. So the **absolute maximum** is 10 at $x = -1$, and the **absolute minimum** is -22 at $x = 3$. Notice that in this case neither endpoint produced the extreme value — but that will not always be true, as the next pitfall emphasizes.

(!) Lỗi thường gặp: Lỗi cực kỳ phổ biến: chỉ tìm cực trị **địa phương** bằng First Derivative Test rồi dừng lại, **quên không kiểm tra hai đầu mút**. Trên một đoạn đóng $[a, b]$, giá trị lớn nhất/nhỏ nhất tuyệt đối hoàn toàn có thể nằm ngay tại $x = a$ hoặc $x = b$ — kể cả khi đạo hàm tại đó khác 0. Ví dụ: $f(x) = x^3 - 3x + 1$ trên $[0, 2]$ có $f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$, chỉ một điểm tới hạn $x = 1$ nằm trong $(0, 2)$ với $f(1) = -1$; nhưng $f(2) = 8 - 6 + 1 = 3$ tại đầu mút phải lại **lớn hơn** $f(1)$, nên max tuyệt đối là 3 tại $x = 2$ chứ không phải tại điểm tới hạn.

5.6 Concavity and the Second Derivative Test

5.6.1 Concavity

Knowing where f increases and decreases describes the overall trend of the graph, but not its *shape*. The second derivative captures shape: whether the curve bends upward like a cup, or downward like a dome.

Concavity from f''

Let f be twice differentiable on an open interval I.

- If $f''(x) > 0$ for every x in I, the graph of f is **concave up** on I (it bends upward, like $y = x^2$; the graph lies above its tangent lines, and f' is increasing).
- If $f''(x) < 0$ for every x in I, the graph of f is **concave down** on I (it bends downward, like $y = -x^2$; the graph lies below its tangent lines, and f' is decreasing).

VN **GIẢI THÍCH TIẾNG VIỆT**

Tính lồi–lõm (concavity) được xác định bởi dấu của f'' — đạo hàm của f'. $f''(x) > 0$ trên một khoảng $\Rightarrow$ đồ thị **lõm lên (concave up)**, giống hình chữ U, và f' đang **tăng** (đồ thị "dốc" ngày càng nhanh theo chiều dương). $f''(x) < 0 \Rightarrow$ đồ thị **lõm xuống (concave down)**, giống mái vòm, và f' đang **giảm**. Hãy nhớ: lồi–lõm nói về **đạo hàm bậc nhất đang tăng hay giảm**, không phải bản thân f đang tăng hay giảm.

5.6.2 The Second Derivative Test

Concavity gives a second — often faster — way to classify a critical point where $f'(c) = 0$.

Second Derivative Test. Suppose $f'(c) = 0$ and f'' exists near c.

- If $f''(c) > 0$, the graph is concave up at c, so f has a **local minimum** at c.
- If $f''(c) < 0$, the graph is concave down at c, so f has a **local maximum** at c.
- If $f''(c) = 0$, the test is **inconclusive** — it gives no information either way, and the **First Derivative Test must be used instead**.

VN **GIẢI THÍCH TIẾNG VIỆT**

Quy tắc Đạo hàm Bậc hai (Second Derivative Test) áp dụng khi $f'(c) = 0$: nếu $f''(c) > 0$ (lõm lên tại đó) $\Rightarrow$ **cực tiểu**; nếu $f''(c) < 0$ (lõm xuống) $\Rightarrow$ **cực đại**. Nhưng nếu $f''(c) = 0$, quy tắc này **không cho kết luận gì cả** (inconclusive) — **bắt buộc** phải quay lại dùng First Derivative Test (xét dấu f' hai bên c) để xác định.

Ví dụ mẫu · Second Derivative Test (successful case)

Classify the critical points of $f(x) = x^3 - 6x^2 + 9x$.

$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3)$, so critical points are $x = 1$ and $x = 3$.

$f''(x) = 6x - 12$.

$$f''(1) = 6 - 12 = -6 < 0 \;\Rightarrow\; \text{local } \textbf{maximum} \text{ at } x = 1, \qquad f(1) = 1 - 6 + 9 = 4.$$

$$f''(3) = 18 - 12 = 6 > 0 \;\Rightarrow\; \text{local } \textbf{minimum} \text{ at } x = 3, \qquad f(3) = 27 - 54 + 27 = 0.$$

So f has a local maximum $(1, 4)$ and a local minimum $(3, 0)$.

Ví dụ mẫu · Second Derivative Test (inconclusive case)

Analyze the critical points of $g(x) = x^4 - 4x^3$.

$g'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$, so critical points are $x = 0$ and $x = 3$.

$g''(x) = 12x^2 - 24x = 12x(x-2)$.

$$g''(3) = 12(3)(1) = 36 > 0 \;\Rightarrow\; \text{local } \textbf{minimum} \text{ at } x = 3, \qquad g(3) = 81 - 108 = -27.$$

$$g''(0) = 12(0)(-2) = 0 \;\Rightarrow\; \textbf{inconclusive at } x = 0.$$

Since the Second Derivative Test fails at $x = 0$, fall back on the First Derivative Test. Because $g'(x) = 4x^2(x-3)$ and $x^2 \geq 0$ always, the sign of g' matches the sign of $(x-3)$ — which is **negative for every** x **near** 0 (both slightly less than and slightly greater than 0, since $0 < 3$). g' does **not** change sign at $x = 0$, so $x = 0$ is **not** a local extremum — it is a critical point where the graph merely has a momentary horizontal tangent while continuing to decrease. (Section 5.7 will show that $x = 0$ is, however, a genuine point of inflection.)

(!) Lỗi thường gặp: Nếu $f''(c) = 0$, **tuyệt đối không được** kết luận "không phải cực trị" hay đoán bừa max/min. $f''(c) = 0$ chỉ có nghĩa Second Derivative Test **bất lực** tại điểm đó — nó không nói lên được điều gì về bản chất của điểm c. Bắt buộc quay lại First Derivative Test (xét dấu f' ở hai bên c) để có kết luận chắc chắn. Ví dụ $g(x) = x^4 - 4x^3$ ở trên: $g''(0) = 0$ nhưng $x = 0$ hoàn toàn không phải cực trị (vì g' không đổi dấu) — nếu chỉ dựa vào $f''(0) = 0$ mà đoán ngẫu nhiên, kết luận rất dễ sai.

5.7 Points of Inflection

Point of inflection

A point $(c, f(c))$ is a **point of inflection** of the graph of f if f is continuous at c and the concavity of f **actually changes** — from concave up to concave down, or from concave down to concave up — as x passes through c.

Because concavity is governed by the sign of f'', a necessary condition for an inflection point at c is that $f''(c) = 0$ or $f''(c)$ is undefined. But — exactly parallel to the critical-point situation in Section 5.1 — this is a **necessary**, not a **sufficient**, condition. The only way to confirm a genuine inflection point is to verify that f'' actually changes sign on either side of c, using the same test-point method as a first-derivative sign chart.

VN **GIẢI THÍCH TIẾNG VIỆT**

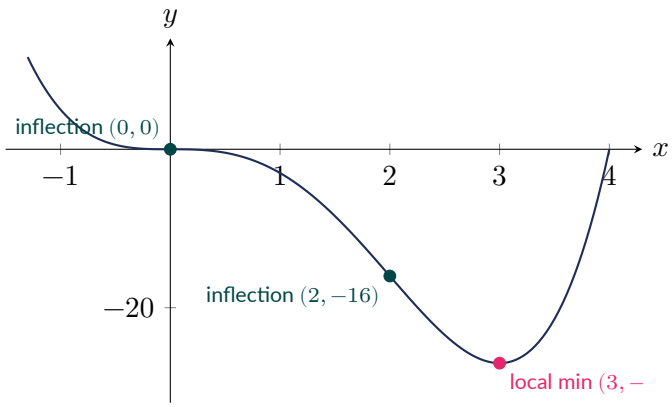
Điểm uốn (inflection point) là nơi **tính lồi–lõm thực sự đổi chiều**. Điều kiện cần là $f''(c) = 0$ (hoặc không xác định), nhưng đây **không phải** điều kiện đủ — hoàn toàn giống tình huống "điểm tới hạn chưa chắc là cực trị". Phải kiểm tra f'' có **thực sự đổi dấu** hai bên c hay không bằng cách thử điểm, giống hệt cách lập bảng xét dấu cho f'.

Returning to $g(x) = x^4 - 4x^3$ from the previous section: $g''(x) = 12x(x-2)$, which is zero at $x = 0$ and $x = 2$. Testing points in the three resulting intervals:

$$g''(-1) = 12(-1)(-3) = 36 > 0 \text{ (concave up)}, \qquad g''(1) = 12(1)(-1) = -12 < 0 \text{ (concave down)},$$

$$g''(3) = 12(3)(1) = 36 > 0 \text{ (concave up)}.$$

The sign of g'' genuinely changes at *both* $x = 0$ (from $+$ to $-$) and $x = 2$ (from $-$ to $+$), so **both** are points of inflection: $(0, 0)$ and $(2, g(2)) = (2, -16)$. Notice that $x = 0$ is simultaneously a critical point (horizontal tangent) that is *not* a local extremum, *and* a point of inflection — a single x-value can carry both roles.



$g(x) = x^4 - 4x^3$: a local minimum at $x = 3$ (confirmed by the Second Derivative Test), and inflection points at $x = 0$ and $x = 2$ where the concavity genuinely switches sign. Note $x = 0$ is a critical point but *not* an extremum.

(!) Lỗi thường gặp: $f''(c) = 0$ **không tự động** nghĩa là c là điểm uốn — phải kiểm tra đổi dấu thật sự. Ví dụ kinh điển: $g(x) = x^4$ có $g''(x) = 12x^2$, và $g''(0) = 0$. Nhưng $g''(x) = 12x^2 \geq 0$ với **mọi** x (không bao giờ âm) — tức f'' **không hề đổi dấu** tại $x = 0$, đồ thị luôn lõm lên (hoặc phẳng tức thời) ở cả hai phía. Vậy $x = 0$ **không phải** điểm uốn của x^4, mặc dù $g''(0) = 0$ — nó thực ra là một điểm cực tiểu (kiểm tra bằng First Derivative Test vì Second Derivative Test cũng vô hiệu ở đây).

5.8 Curve Sketching: Connecting a Function and Its Derivatives

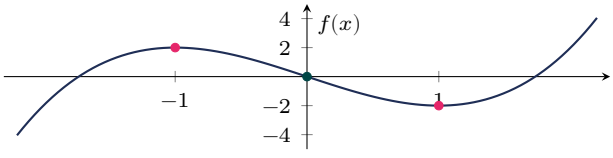
5.8.1 Reading Derivative Graphs

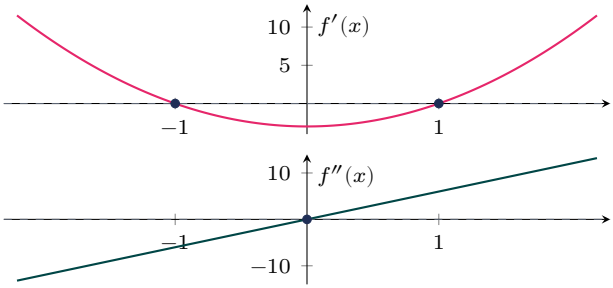
Every fact established so far can be read in either direction. Given a formula (or graph) for f, we can deduce the shape of f' and f''; and given information about f' or f'' (a sign chart, a table, or a graph), we can deduce the shape of f itself. AP exam questions frequently give the graph of f' (*not* f) and ask about the behavior of f — so it is essential to internalize the translation table below.

$f' > 0$ on an interval	f is increasing there
$f' < 0$ on an interval	f is decreasing there
f' changes $+ \to -$ at c	local maximum of f at c
f' changes $- \to +$ at c	local minimum of f at c
$f'' > 0$ on an interval	f is concave up there (f' increasing)
$f'' < 0$ on an interval	f is concave down there (f' decreasing)
f'' changes sign at c	point of inflection of f at c
$f''(c) = 0$ only, no sign change	*not* an inflection point — verify first!

VN **GIẢI THÍCH TIẾNG VIỆT**

Bảng trên là "từ điển" chuyển đổi hai chiều giữa f, f', và f''. Đề thi AP rất hay cho **đồ thị của** f' (không phải f) rồi hỏi về hành vi của f: hãy đọc **dấu** và **sự đổi dấu** của đồ thị f' (nó nằm trên hay dưới trục hoành, và có cắt trục hoành hay không) để suy ra f đang tăng/giảm và có cực trị ở đâu; tương tự, đọc xem đồ thị f' đang **tăng hay giảm** (chứ không phải nằm trên/dưới trục) để suy ra tính lồi–lõm của f.





Top to bottom, on the same x-scale: $f(x) = x^3 - 3x$, $f'(x) = 3x^2 - 3$, $f''(x) = 6x$. Where $f' > 0$ (outside $[-1, 1]$), f is increasing; where $f' < 0$ (inside $(-1, 1)$), f is decreasing; $f' = 0$ at the local max/min $x = \pm 1$. Where $f'' < 0$ ($x < 0$), f is concave down; where $f'' > 0$ ($x > 0$), concave up; the sign change of f'' at $x = 0$ marks the inflection point.

5.8.2 A Complete Curve Sketch

The following example walks through a full analysis, combining every tool from this chapter: domain, symmetry, intercepts, first-derivative behavior, second-derivative behavior, and asymptotes.

Ví dụ mẫu · Full curve analysis

Analyze and sketch $f(x) = \dfrac{1}{x^2 + 1}$.

Domain and symmetry: $x^2 + 1 > 0$ for every real x, so the domain is all reals. Since $f(-x) = \dfrac{1}{(-x)^2 + 1} = f(x)$, f is **even** — its graph is symmetric about the y-axis.

Intercepts: $f(0) = \dfrac{1}{0 + 1} = 1$, so the y-intercept is $(0, 1)$. Since the numerator 1 is never zero, $f(x)$ is never zero — there are **no x-intercepts**.

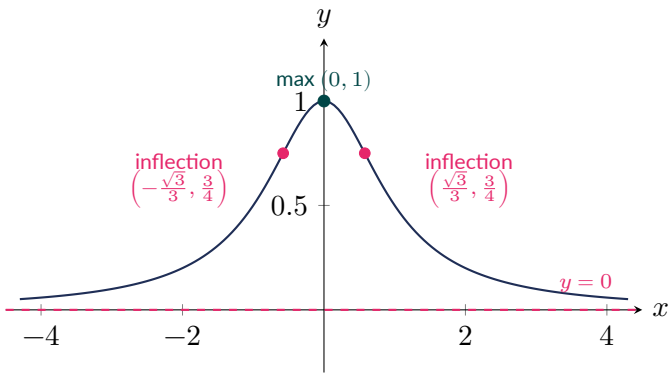
First derivative: $f'(x) = -2x(x^2 + 1)^{-2} = \dfrac{-2x}{(x^2 + 1)^2}$. Since $(x^2 + 1)^2 > 0$ always, the only critical point is $x = 0$ (from the numerator). For $x < 0$, $-2x > 0$ so $f' > 0$ (increasing); for $x > 0$, $-2x < 0$ so $f' < 0$ (decreasing). f' changes $+ \to -$ at $x = 0$, so f has a local (in fact absolute) **maximum** at $(0, 1)$.

Second derivative: differentiating $f'(x) = -2x(x^2 + 1)^{-2}$ with the product rule,

$$f''(x) = -2(x^2 + 1)^{-2} + (-2x)(-2)(x^2 + 1)^{-3}(2x) = \frac{-2(x^2 + 1) + 8x^2}{(x^2 + 1)^3} = \frac{6x^2 - 2}{(x^2 + 1)^3}.$$

Setting $6x^2 - 2 = 0$ gives $x^2 = \frac{1}{3}$, so $x = \pm\dfrac{1}{\sqrt{3}} = \pm\dfrac{\sqrt{3}}{3}$. Since $(x^2 + 1)^3 > 0$ always, the sign of f'' matches the sign of $6x^2 - 2$: negative (concave down) for $|x| < \frac{1}{\sqrt{3}}$, positive (concave up) for $|x| > \frac{1}{\sqrt{3}}$ — a genuine sign change at both roots. So f has **inflection points** at $x = \pm\dfrac{\sqrt{3}}{3}$, where $f\left(\pm\dfrac{\sqrt{3}}{3}\right) = \dfrac{1}{\frac{1}{3} + 1} = \dfrac{3}{4}$.

Asymptotes: as $x \to \pm\infty$, $f(x) \to 0$, so $y = 0$ is a **horizontal asymptote**. The denominator $x^2 + 1$ is never zero, so there is **no vertical asymptote**.



$f(x) = \dfrac{1}{x^2+1}$ (the Witch of Agnesi): even, no x-intercepts, absolute maximum $(0, 1)$, inflection points at $x = \pm\frac{\sqrt{3}}{3} \approx \pm 0.577$, horizontal asymptote $y = 0$, no vertical asymptote.

VN GIẢI THÍCH TIẾNG VIỆT

Quy trình phác họa đồ thị đầy đủ luôn theo thứ tự: (1) miền xác định và tính đối xứng (chẵn/lẻ); (2) giao điểm với hai trục; (3) f' để tìm điểm tới hạn, khoảng đồng biến/nghịch biến, và cực trị; (4) f'' để tìm khoảng lồi/lõm và điểm uốn; (5) tiệm cận (đứng và ngang). Ví dụ $f(x) = \frac{1}{x^2+1}$ ở trên minh họa đầy đủ cả năm bước — kể cả trường hợp "không có" (không có giao điểm với trục hoành, không có tiệm cận đứng) cũng là thông tin quan trọng cần nêu rõ.

5.9 Optimization Problems

5.9.1 Setting Up Optimization Problems

Optimization problems ask for the largest or smallest possible value of some quantity, subject to a constraint. The calculus is usually the easy part; the hard part is translating a word problem into a single-variable function.

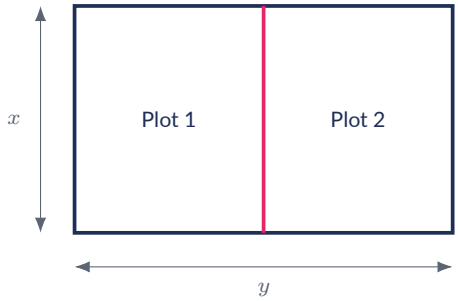
General optimization procedure.

1. **Identify** the quantity to be maximized or minimized, and write it as a function of the relevant variables (the **objective function**).
2. **Identify the constraint** — a fixed relationship among the variables (fixed perimeter, fixed volume, a point restricted to lie on a curve, etc.) — and use it to eliminate all but **one** variable in the objective function.
3. **Determine the domain** of the resulting single-variable function from the physical context (e.g., lengths must be positive).
4. **Find critical points** of the objective function by setting its derivative to zero (or finding where it is undefined).
5. **Justify** that the critical point actually gives the desired maximum or minimum — using the First Derivative Test, the Second Derivative Test, or (on a closed domain) the Candidates Test, comparing against the domain's endpoints/limits.
6. **Answer the question actually asked** — often a dimension or a maximal/minimal value, not just the critical x-value.

VN GIẢI THÍCH TIẾNG VIỆT

Quy trình giải bài toán tối ưu (optimization) luôn gồm các bước: (1) viết **hàm mục tiêu** (objective function) — đại lượng cần max/min — theo các biến; (2) dùng **ràng buộc** (constraint) để rút một biến, đưa hàm mục tiêu về **một biến duy nhất**; (3) xác định miền xác định hợp lý theo ngữ cảnh

thực tế (độ dài phải dương, v.v.); (4) tìm điểm tới hạn bằng đạo hàm; (5) **bắt buộc chứng minh** điểm đó thực sự là max/min (bằng First/Second Derivative Test hoặc so sánh với biên miền) — đây là bước học sinh hay bỏ qua nhất nhưng lại chiếm điểm quan trọng trong bài thi tự luận.



A rectangular region of dimensions x by y, divided into two equal plots by an internal dividing fence (pink) parallel to the sides of length x. Total fencing needed equals the outer perimeter plus this internal divider.

5.9.2 Optimization Worked Examples

Ví dụ mẫu · Minimizing surface area for fixed volume

A closed cylindrical can must have volume 16π cm^3. Find the radius and height that minimize the total surface area (top, bottom, and lateral surface), and find that minimum area.

Objective: surface area $S = 2\pi r^2 + 2\pi rh$ (two circular ends plus the lateral surface).

Constraint: volume $V = \pi r^2 h = 16\pi$, so $h = \dfrac{16}{r^2}$.

Reduce to one variable:

$$S(r) = 2\pi r^2 + 2\pi r \cdot \frac{16}{r^2} = 2\pi r^2 + \frac{32\pi}{r}, \qquad r > 0.$$

Critical points: $S'(r) = 4\pi r - \dfrac{32\pi}{r^2}$. Setting $S'(r) = 0$:

$$4\pi r = \frac{32\pi}{r^2} \implies 4r^3 = 32 \implies r^3 = 8 \implies r = 2 \text{ cm}.$$

Justify the minimum: $S''(r) = 4\pi + \dfrac{64\pi}{r^3} > 0$ for every $r > 0$, so S is concave up everywhere on its domain — the unique critical point $r = 2$ must be an absolute minimum (also consistent with $S(r) \to \infty$ as $r \to 0^+$ or $r \to \infty$).

Answer: $r = 2$ cm, and $h = \dfrac{16}{4} = 4$ cm (so $h = 2r$). The minimum surface area is

$$S(2) = 2\pi(4) + \frac{32\pi}{2} = 8\pi + 16\pi = 24\pi \text{ cm}^2 \approx 75.4 \text{ cm}^2.$$

Ví dụ mẫu · Maximizing area for fixed fencing

A farmer has 240 m of fencing to build a rectangular pen that is divided into two equal-sized plots by an internal fence parallel to one pair of sides (as in the figure above). Find the dimensions that maximize the total enclosed area, and find that maximum area.

Setup: let x be the length of each of the three fence segments running in one direction (the

two outer sides plus the internal divider, all parallel to each other), and y the length of the other two (undivided) sides. Total fencing used is $3x + 2y$, and the total enclosed area is $A = xy$.

Constraint: $3x + 2y = 240 \Rightarrow y = \dfrac{240 - 3x}{2}$.

Reduce to one variable:

$$A(x) = x \cdot \frac{240 - 3x}{2} = 120x - \frac{3}{2}x^2, \qquad 0 < x < 80.$$

Critical point: $A'(x) = 120 - 3x$. Setting $A'(x) = 0$ gives $x = 40$.

Justify the maximum: $A''(x) = -3 < 0$ for all x, so A is concave down everywhere — the unique critical point is an absolute maximum on the domain (also $A \to 0$ as $x \to 0^+$ or $x \to 80^-$, consistent with an interior maximum).

Answer: $x = 40$ m, and $y = \dfrac{240 - 120}{2} = 60$ m. The maximum area is

$$A(40) = 40 \times 60 = 2400 \text{ m}^2.$$

Ví dụ mẫu · Minimizing distance

Find the point on the parabola $y = x^2$ that is closest to the point $(3, 0)$, and find that minimum distance.

Objective: rather than minimizing distance directly (which involves a square root), minimize the **square** of the distance — it is minimized at the same x-value, since $t \mapsto \sqrt{t}$ is increasing. For a point (x, x^2) on the parabola,

$$D(x) = (x - 3)^2 + (x^2 - 0)^2 = (x - 3)^2 + x^4 = x^4 + x^2 - 6x + 9.$$

Critical points: $D'(x) = 4x^3 + 2x - 6$. Setting $D'(x) = 0$: $4x^3 + 2x - 6 = 0$, i.e. $2x^3 + x - 3 = 0$. Testing $x = 1$: $2(1) + 1 - 3 = 0$ — a root. Factoring,

$$2x^3 + x - 3 = (x - 1)(2x^2 + 2x + 3).$$

The quadratic factor $2x^2 + 2x + 3$ has discriminant $2^2 - 4(2)(3) = 4 - 24 = -20 < 0$, so it has no real roots. Thus $x = 1$ is the **only** critical point.

Justify the minimum: $D''(x) = 12x^2 + 2 > 0$ for every real x, so D is concave up everywhere (strictly convex) — the unique critical point $x = 1$ must give the absolute minimum of D, and hence of the actual distance.

Answer: the closest point is $(1, 1)$, and the minimum distance is

$$\sqrt{D(1)} = \sqrt{(1 - 3)^2 + 1^4} = \sqrt{4 + 1} = \sqrt{5}.$$

(!) Lỗi thường gặp: Tìm được điểm tới hạn trong bài toán tối ưu **chưa phải là xong** — phải **chứng minh** đó thực sự là max hay min theo đúng yêu cầu đề bài, không được ngầm giả định. Cách phổ biến nhất: dùng Second Derivative Test (nếu f'' luôn cùng dấu trên miền xác định thì kết luận ngay là max/min tuyệt đối), hoặc First Derivative Test (lập bảng xét dấu), hoặc so sánh với giá trị tại biên miền (đặc biệt khi hàm mục tiêu không xác định trên toàn trục số, ví dụ độ dài phải dương). Bỏ qua bước này là lỗi bị trừ điểm nhiều nhất trong các bài tự luận tối ưu hóa của đề thi AP.

5.10 Implicit Relations: Tangent Lines and Monotonicity

Not every curve is the graph of a function $y = f(x)$; many important curves (circles, ellipses, and more general relations) are defined **implicitly** by an equation in x and y. Implicit differentiation — treating y as an unspecified function of x and applying the chain rule to every y-term — produces a formula for $\dfrac{dy}{dx}$ that is itself usually a function of **both** x and y. That formula is exactly what is needed to locate horizontal and vertical tangent lines, and to determine where the curve is rising or falling.

Horizontal and vertical tangents on implicit curves

Suppose implicit differentiation produces $\dfrac{dy}{dx} = \dfrac{N(x,y)}{D(x,y)}$ (a ratio in lowest terms).

- A **horizontal tangent** occurs where $N(x,y) = 0$ while $D(x,y) \neq 0$ (slope equals 0).
- A **vertical tangent** occurs where $D(x,y) = 0$ while $N(x,y) \neq 0$ (slope is undefined).
- Where **both** $N = 0$ and $D = 0$ simultaneously, the point requires separate investigation (it is often not even on the curve, or is a singular point).

In both cases, the candidate x- or y-values must be substituted back into the **original** equation of the curve to find the actual point(s), and then checked against the other (numerator or denominator) condition.

VN **GIẢI THÍCH TIẾNG VIỆT**

Với một quan hệ ẩn (implicit relation), sau khi đạo hàm ẩn ta được $\dfrac{dy}{dx} = \dfrac{N(x,y)}{D(x,y)}$. **Tiếp tuyến ngang** (horizontal tangent, hệ số góc $= 0$) xảy ra khi **tử số** $N = 0$ (và mẫu số khác 0). **Tiếp tuyến đứng** (vertical tangent, hệ số góc không xác định) xảy ra khi **mẫu số** $D = 0$ (và tử số khác 0). Sau khi tìm được điều kiện, luôn phải **thay ngược lại phương trình gốc** của đường cong để tìm toạ độ điểm cụ thể — đừng quên bước này.

Ví dụ mẫu · Horizontal and vertical tangents

Find all points where the curve $x^2 - xy + y^2 = 3$ has a horizontal tangent, and all points where it has a vertical tangent.

Implicit differentiation (product rule on the xy term):

$$2x - \left(y + x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0 \implies \frac{dy}{dx}(2y - x) = y - 2x \implies \frac{dy}{dx} = \frac{y - 2x}{2y - x}.$$

Horizontal tangents ($y - 2x = 0$, i.e. $y = 2x$, with $2y - x \neq 0$): substituting into the original equation,

$$x^2 - x(2x) + (2x)^2 = 3 \implies 3x^2 = 3 \implies x = \pm 1.$$

This gives points $(1, 2)$ and $(-1, -2)$. Checking the denominator at each: at $(1, 2)$, $2y - x = 4 - 1 = 3 \neq 0$; at $(-1, -2)$, $2y - x = -4 + 1 = -3 \neq 0$. Both are valid horizontal-tangent points.

Vertical tangents ($2y - x = 0$, i.e. $x = 2y$, with $y - 2x \neq 0$): substituting,

$$(2y)^2 - (2y)y + y^2 = 3 \implies 3y^2 = 3 \implies y = \pm 1.$$

This gives points $(2, 1)$ and $(-2, -1)$. Checking the numerator: at $(2, 1)$, $y - 2x = 1 - 4 = -3 \neq 0$; at $(-2, -1)$, $y - 2x = -1 + 4 = 3 \neq 0$. Both are valid vertical-tangent points.

Summary: horizontal tangents at $(1, 2)$ and $(-1, -2)$; vertical tangents at $(2, 1)$ and $(-2, -1)$.

The same formula for $\frac{dy}{dx}$ also tells us where the curve is rising or falling at any specific point on it — simply substitute the coordinates and read the sign, exactly as with an explicit function.

5.11 Problem Bank

Bài tập luyện tập

Explain whether the Extreme Value Theorem guarantees an absolute maximum and absolute minimum for $g(x) = \dfrac{1}{x-2}$ on the interval $[0, 1]$, and separately for $h(x) = \dfrac{1}{x-2}$ on the interval $[1, 3]$.

Lời giải chi tiết

On $[0, 1]$, the value $x = 2$ (where g is undefined) does not lie in the interval, so g is continuous on all of $[0, 1]$. Since $[0, 1]$ is closed and bounded, the EVT applies: g is guaranteed to attain both an absolute maximum and an absolute minimum on $[0, 1]$. On $[1, 3]$, however, $x = 2$ *does* lie inside the interval, and $h(x) = \frac{1}{x-2}$ has a vertical asymptote there — h is **not continuous** on $[1, 3]$, so the EVT's hypotheses fail, and it gives no guarantee. Indeed, $h(x) \to +\infty$ as $x \to 2^-$ and $h(x) \to -\infty$ as $x \to 2^+$, so h has neither an absolute maximum nor an absolute minimum on $[1, 3]$.

Bài tập luyện tập

How many critical points does $f(x) = 2x^3 + 3x^2 - 12x + 7$ have?

(A) 0
(B) 1
(C) 2
(D) 3

Lời giải chi tiết

$f'(x) = 6x^2 + 6x - 12 = 6(x^2 + x - 2) = 6(x + 2)(x - 1)$, which is defined for all x and equals zero at $x = -2$ and $x = 1$. Both are in the domain, so f has exactly 2 critical points. **(C)**.

Bài tập luyện tập

What is the critical point of $f(x) = (x + 3)^{2/3} - 1$?

(A) $x = 3$
(B) $x = -3$
(C) $x = 0$
(D) $x = -1$

Lời giải chi tiết

$f'(x) = \dfrac{2}{3}(x + 3)^{-1/3} = \dfrac{2}{3\sqrt[3]{x + 3}}$, which is undefined at $x = -3$. Since $f(-3) = 0^{2/3} - 1 = -1$ is defined, $x = -3$ is in the domain, so it is a critical point (a cusp). **(B)**.

Bài tập luyện tập

Verify that Rolle's Theorem applies to $f(x) = \sin x$ on $[0, \pi]$, and find the value of c it guarantees.

Lời giải chi tiết

$f(x) = \sin x$ is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$ (sine is differentiable everywhere), and $f(0) = \sin 0 = 0$, $f(\pi) = \sin \pi = 0$, so $f(0) = f(\pi)$. All hypotheses hold. Since $f'(x) = \cos x$, solving $\cos c = 0$ on $(0, \pi)$ gives $c = \dfrac{\pi}{2}$.

Bài tập luyện tập

Let $f(x) = x^3 - 4x$ on $[0, 3]$. Find the value(s) of c guaranteed by the Mean Value Theorem.

Lời giải chi tiết

f is a polynomial, so it is continuous on $[0, 3]$ and differentiable on $(0, 3)$; the MVT applies. $f(0) = 0$, $f(3) = 27 - 12 = 15$, so the average rate of change is $\dfrac{15 - 0}{3 - 0} = 5$. Since $f'(x) = 3x^2 - 4$, solve $3c^2 - 4 = 5 \Rightarrow 3c^2 = 9 \Rightarrow c^2 = 3 \Rightarrow c = \pm\sqrt{3}$. Only $c = \sqrt{3} \approx 1.73$ lies in $(0, 3)$, so $c = \sqrt{3}$.

Bài tập luyện tập

Let $g(x) = 2x^3 - 3x^2 - 12x + 1$. On which interval is g decreasing?

(A) $(-\infty, -1)$

(B) $(-1, 2)$

(C) $(2, \infty)$

(D) g is never decreasing

Lời giải chi tiết

$g'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x - 2)(x + 1)$, zero at $x = -1, 2$. Testing $x = 0$: $g'(0) = 6(-2)(1) = -12 < 0$, so g is decreasing on $(-1, 2)$; testing $x = -2$ and $x = 3$ both give $g' > 0$, confirming g increases outside $[-1, 2]$. **(B)**.

Bài tập luyện tập

Let $h(x) = x^4 - 8x^2 + 3$. Construct the sign chart for h', and classify every critical point as a local maximum, local minimum, or neither.

Lời giải chi tiết

$h'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x - 2)(x + 2)$, with critical points $x = -2, 0, 2$. Testing points:

$$h'(-3) = 4(-3)(5)(-1) = -60 < 0, \quad h'(-1) = 4(-1)(-1)(3)?$$

Recomputing carefully: $h'(-1) = 4(-1)((-1) - 2)((-1) + 2) = 4(-1)(-3)(1) = 12 > 0$; $h'(1) = 4(1)(1 - 2)(1 + 2) = 4(1)(-1)(3) = -12 < 0$; $h'(3) = 4(3)(1)(5) = 60 > 0$.

Interval	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, \infty)$
Sign of h'	$-$	$+$	$-$	$+$

h' changes $- \to +$ at $x = -2$: **local minimum**, $h(-2) = 16 - 32 + 3 = -13$. h' changes $+ \to -$ at $x = 0$: **local maximum**, $h(0) = 3$. h' changes $- \to +$ at $x = 2$: **local minimum**, $h(2) = 16 - 32 + 3 = -13$.

Bài tập luyện tập

At which x-value does $f(x) = xe^{-x}$ have a local maximum?

(A) $x = 0$
(B) $x = 1$
(C) $x = e$
(D) f has no local maximum

Lời giải chi tiết

$f'(x) = e^{-x}(1 - x)$, zero only at $x = 1$ (since $e^{-x} > 0$ always); $f' > 0$ for $x < 1$ and $f' < 0$ for $x > 1$, so f has a local maximum at $x = 1$. **(B).**

Bài tập luyện tập

Show that $x = 2$ is a critical point of $f(x) = (x - 2)^3 + 1$, but that it is *not* a local extremum. Use concavity to describe what kind of point it is instead.

Lời giải chi tiết

$f'(x) = 3(x-2)^2$, so $f'(2) = 0$ and $x = 2$ is a critical point. Since $(x-2)^2 \geq 0$ for all x, $f'(x) \geq 0$ everywhere — in particular f' is positive on both sides of $x = 2$ (it never becomes negative), so f' does **not** change sign there, and $x = 2$ is **not** a local extremum (f is increasing right through that point). Checking concavity: $f''(x) = 6(x-2)$, which is negative for $x < 2$ and positive for $x > 2$ — a genuine sign change. So $x = 2$ is a point of **inflection** (with a momentarily horizontal tangent).

Bài tập luyện tập

Find the absolute maximum and absolute minimum of $f(x) = x^4 - 2x^2 + 3$ on $[-2, 2]$.

Lời giải chi tiết

$f'(x) = 4x^3-4x = 4x(x-1)(x+1)$, with critical points $x = -1, 0, 1$, all inside $(-2, 2)$. Evaluate:

$$f(-2) = 16 - 8 + 3 = 11,\ f(-1) = 1 - 2 + 3 = 2,\ f(0) = 3,\ f(1) = 2,\ f(2) = 11.$$

Comparing all five values, the largest is 11 (attained at both $x = \pm 2$) and the smallest is 2 (attained at both $x = \pm 1$). Absolute maximum $= 11$; absolute minimum $= 2$.

Bài tập luyện tập

Let $f(x) = x^3 - 3x + 1$ on $[0, 2]$. The absolute maximum value of f on this interval is:

(A) -1
(B) 1
(C) 3
(D) cannot be determined

Lời giải chi tiết

$f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$; the only critical point inside $(0, 2)$ is $x = 1$. Candidates: $f(0) = 1$, $f(1) = 1 - 3 + 1 = -1$, $f(2) = 8 - 6 + 1 = 3$. The largest value is $f(2) = 3$, occurring at the **endpoint** $x = 2$, not at the critical point. **(C).**

Bài tập luyện tập

For $f(x) = x^4 - 6x^2$, the Second Derivative Test classifies the critical point $x = 0$ as a:

(A) Local maximum
(B) Local minimum
(C) Inflection point
(D) Point where the test is inconclusive

Lời giải chi tiết

$f'(x) = 4x^3 - 12x = 4x(x^2 - 3)$, critical points $x = 0, \pm\sqrt{3}$. $f''(x) = 12x^2 - 12$, so $f''(0) = -12 < 0$, which by the Second Derivative Test means $x = 0$ is a **local maximum**. **(A)**.

Bài tập luyện tập

Let $f(x) = x^5$. Show that the Second Derivative Test is inconclusive at the critical point $x = 0$, then use the First Derivative Test to determine whether $x = 0$ is a local extremum.

Lời giải chi tiết

$f'(x) = 5x^4$, so $f'(0) = 0$ and $x = 0$ is a critical point. $f''(x) = 20x^3$, so $f''(0) = 0$ — the Second Derivative Test is inconclusive. Using the First Derivative Test instead: $f'(x) = 5x^4 \geq 0$ for **every** real x (an even power), so f' does not change sign at $x = 0$ — f is increasing on both sides, and $x = 0$ is **not** a local extremum. (Checking concavity confirms $f''(x) = 20x^3$ changes from negative to positive at $x = 0$, so $x = 0$ is instead a point of inflection.)

Bài tập luyện tập

Which statement correctly describes the behavior of $g(x) = x^4$ at $x = 0$?

(A) $x = 0$ is an inflection point because $g''(0) = 0$
(B) $x = 0$ is not an inflection point because g'' does not change sign there
(C) $x = 0$ is a local maximum
(D) g is concave down on an interval containing 0

Lời giải chi tiết

$g''(x) = 12x^2 \geq 0$ for every x, so g'' never becomes negative — it does not change sign at $x = 0$, so $x = 0$ is **not** an inflection point despite $g''(0) = 0$. (Separately, $g'(x) = 4x^3$ changes from negative to positive at $x = 0$, so $x = 0$ is actually a local minimum.) **(B)**.

Bài tập luyện tập

For $f(x) = x^3 - 6x^2 + 9x + 1$: find all local extrema (using the Second Derivative Test) and all points of inflection, and describe the concavity of f on each relevant interval.

Lời giải chi tiết

$f'(x) = 3x^2 - 12x + 9 = 3(x - 1)(x - 3)$, critical points $x = 1, 3$. $f''(x) = 6x - 12$. Since $f''(1) = -6 < 0$, $x = 1$ is a **local maximum**, $f(1) = 1 - 6 + 9 + 1 = 5$. Since $f''(3) = 6 > 0$, $x = 3$ is a **local minimum**, $f(3) = 27 - 54 + 27 + 1 = 1$.

For inflection: $f''(x) = 6x - 12 = 0$ at $x = 2$; since f'' is linear and increasing, $f'' < 0$ for $x < 2$

(concave down) and $f'' > 0$ for $x > 2$ (concave up) — a genuine sign change, so $(2, f(2)) = (2, 3)$ is a point of inflection. Summary: concave down on $(-\infty, 2)$, concave up on $(2, \infty)$, local max $(1, 5)$, local min $(3, 1)$, inflection point $(2, 3)$.

Bài tập luyện tập

Let $f(x) = x^4 - 4x^3 + 4x^2$. (a) Find all critical points and classify each with the First Derivative Test. (b) Find all points of inflection, in exact form.

Lời giải chi tiết

Note $f(x) = x^2(x-2)^2$.

(a) $f'(x) = 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2) = 4x(x-1)(x-2)$, critical points $x = 0, 1, 2$. Testing: $f'(-1) = 4(-1)(-2)(-3) = -24 < 0$; $f'(0.5) = 4(0.5)(-0.5)(-1.5) = 1.5 > 0$; $f'(1.5) = 4(1.5)(0.5)(-0.5) = -1.5 < 0$; $f'(3) = 4(3)(2)(1) = 24 > 0$. So f decreases, then increases, then decreases, then increases: **local minimum** at $x = 0$ ($f(0) = 0$), **local maximum** at $x = 1$ ($f(1) = 1$), **local minimum** at $x = 2$ ($f(2) = 0$).

(b) $f''(x) = 12x^2 - 24x + 8 = 4(3x^2 - 6x + 2)$. Solving $3x^2 - 6x + 2 = 0$: $x = \dfrac{6 \pm \sqrt{36 - 24}}{6} = \dfrac{6 \pm 2\sqrt{3}}{6} = 1 \pm \dfrac{\sqrt{3}}{3}$. Since $3x^2 - 6x + 2$ is an upward parabola in x, $f'' < 0$ between the roots and $f'' > 0$ outside — a genuine sign change at both, so both are inflection points. Writing $x = 1 + t$ with $t = \pm\frac{\sqrt{3}}{3}$ (so $t^2 = \frac{1}{3}$): $x(x-2) = (1+t)(t-1) = t^2 - 1 = \frac{1}{3} - 1 = -\frac{2}{3}$, so $f(x) = [x(x-2)]^2 = \frac{4}{9}$ at *both* roots. Inflection points: $\left(1 - \frac{\sqrt{3}}{3}, \frac{4}{9}\right)$ and $\left(1 + \frac{\sqrt{3}}{3}, \frac{4}{9}\right)$.

Bài tập luyện tập

The sign of $f'(x)$ is given on four intervals: positive on $(-\infty, -2)$, negative on $(-2, 1)$, positive on $(1, 4)$, negative on $(4, \infty)$. Which x-value is a local minimum of f?

(A) $x = -2$

(B) $x = 1$

(C) $x = 4$

(D) f has no local minimum

Lời giải chi tiết

A local minimum requires f' to change from negative to positive. That happens only at $x = 1$ (negative on $(-2, 1)$, positive on $(1, 4)$). At $x = -2$, f' changes $+ \to -$ (local max); at $x = 4$, f' changes $+ \to -$ (local max). **(B)**.

Bài tập luyện tập

The table below gives the sign of $f'(x)$ and $f''(x)$ on four intervals determined by the values $A < B < C$.

Interval	$(-\infty, A)$	(A, B)	(B, C)	(C, ∞)
Sign of f'	$+$	$+$	$-$	$-$
Sign of f''	$+$	$-$	$-$	$+$

Describe the behavior of f (increasing/decreasing, concave up/down) on each interval, and identify which of A, B, C are local extrema and which are points of inflection.

Lời giải chi tiết

On $(-\infty, A)$: f increasing, concave up. On (A, B): f increasing, concave down. On (B, C): f decreasing, concave down. On (C, ∞): f decreasing, concave up.

At $x = A$: f' stays positive on both sides (no sign change) $\Rightarrow$ **not** a local extremum; but f'' changes $+ \to - \Rightarrow A$ **is** a point of inflection. At $x = B$: f' changes $+ \to - \Rightarrow$ **local maximum**; f'' stays negative on both sides $\Rightarrow$ **not** an inflection point. At $x = C$: f' stays negative on both sides $\Rightarrow$ **not** a local extremum; f'' changes $- \to + \Rightarrow C$ **is** a point of inflection.

Bài tập luyện tập

If $f''(x) > 0$ for every x in (a, b), then on (a, b) the graph of f is:

(A) Increasing
(B) Decreasing
(C) Concave up
(D) Concave down

Lời giải chi tiết

By definition, $f'' > 0$ on an interval means the graph is concave up there. (Concavity says nothing by itself about whether f is increasing or decreasing.) **(C)**.

Bài tập luyện tập

An open-top box with a square base must have volume 64 ft^3. The base material costs \$4 per square foot and the (four) side material costs \$2 per square foot. Find the dimensions that minimize the total cost, and find that minimum cost.

Lời giải chi tiết

Let x be the side of the square base and h the height. Volume: $x^2 h = 64 \Rightarrow h = \dfrac{64}{x^2}$. Cost: base costs $4x^2$; the four sides have combined area $4xh$, costing $2(4xh) = 8xh$. Total cost

$$C(x) = 4x^2 + 8xh = 4x^2 + 8x \cdot \frac{64}{x^2} = 4x^2 + \frac{512}{x}, \qquad x > 0.$$

$C'(x) = 8x - \dfrac{512}{x^2}$. Setting $C'(x) = 0$: $8x^3 = 512 \Rightarrow x^3 = 64 \Rightarrow x = 4$ ft. Then $h = \dfrac{64}{16} = 4$ ft (the box is a cube). Since $C''(x) = 8 + \dfrac{1024}{x^3} > 0$ for all $x > 0$, this critical point is an absolute minimum. Minimum cost:

$$C(4) = 4(16) + \frac{512}{4} = 64 + 128 = \$192.$$

Bài tập luyện tập

A rectangle has its base on the x-axis and its two upper corners on the parabola $y = 9 - x^2$. Find the dimensions of the rectangle with maximum area, and find that maximum area.

Lời giải chi tiết

By symmetry, let the upper corners be $(-x, 9-x^2)$ and $(x, 9-x^2)$ for $0 < x < 3$. Width $= 2x$, height $= 9 - x^2$, so

$$A(x) = 2x(9 - x^2) = 18x - 2x^3.$$

$A'(x) = 18 - 6x^2$. Setting $A'(x) = 0$: $x^2 = 3 \Rightarrow x = \sqrt{3}$ (rejecting the negative root, since $x > 0$). $A''(x) = -12x$, and $A''(\sqrt{3}) = -12\sqrt{3} < 0$, confirming a maximum (also $A(0) = A(3) = 0$ at the domain's edges, consistent with an interior maximum). Width $= 2\sqrt{3}$, height $= 9 - 3 = 6$. Maximum area:

$$A(\sqrt{3}) = 18\sqrt{3} - 2(3\sqrt{3}) = 18\sqrt{3} - 6\sqrt{3} = 12\sqrt{3} \approx 20.8.$$

Bài tập luyện tập

A rectangular garden of area 150 m^2 is to be divided into two equal-size plots by a fence parallel to one pair of sides (as in the earlier figure). Find the dimensions that minimize the total length of fencing used (outer perimeter plus the dividing fence), and find that minimum length.

Lời giải chi tiết

Let x be the length of each of the three parallel fence segments (two outer sides plus the divider) and y the length of the other two sides. Area constraint: $xy = 150 \Rightarrow y = \dfrac{150}{x}$. Total fence:

$$F(x) = 3x + 2y = 3x + \frac{300}{x}, \qquad x > 0.$$

$F'(x) = 3 - \dfrac{300}{x^2}$. Setting $F'(x) = 0$: $3x^2 = 300 \Rightarrow x^2 = 100 \Rightarrow x = 10$ m. Then $y = \dfrac{150}{10} = 15$ m. Since $F''(x) = \dfrac{600}{x^3} > 0$ for $x > 0$, this is an absolute minimum. Minimum fencing:

$$F(10) = 3(10) + \frac{300}{10} = 30 + 30 = 60 \text{ m}.$$

Bài tập luyện tập

Two positive numbers have a product of 100. The minimum possible value of their sum is:

(A) 10 **(C)** 25

(B) 20 **(D)** 50

Lời giải chi tiết

Let the numbers be x and $\dfrac{100}{x}$, $x > 0$. Sum $S(x) = x + \dfrac{100}{x}$. $S'(x) = 1 - \dfrac{100}{x^2} = 0 \Rightarrow x^2 = 100 \Rightarrow x = 10$, giving the other number also 10. $S''(x) = \dfrac{200}{x^3} > 0$ for $x > 0$, confirming a minimum. Minimum sum $= 10 + 10 = 20$. **(B).**

Bài tập luyện tập

An open-top box with a square base must have volume 4000 cm^3. Find the dimensions that minimize the total surface area (base plus four sides), and find that minimum surface area.

Lời giải chi tiết

Let x be the base side and h the height, with $x^2h = 4000 \Rightarrow h = \dfrac{4000}{x^2}$. Surface area: $S(x) = x^2 + 4xh = x^2 + \dfrac{16000}{x}$, $x > 0$. $S'(x) = 2x - \dfrac{16000}{x^2}$. Setting $S'(x) = 0$: $2x^3 = 16000 \Rightarrow x^3 = 8000 \Rightarrow x = 20$ cm. Then $h = \dfrac{4000}{400} = 10$ cm. $S''(x) = 2 + \dfrac{32000}{x^3} > 0$ for $x > 0$, confirming a minimum. Minimum area:

$$S(20) = 400 + \frac{16000}{20} = 400 + 800 = 1200 \text{ cm}^2.$$

Bài tập luyện tập

For the circle $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$ at the point $(3, 4)$.

(A) $-\dfrac{3}{4}$

(B) $\dfrac{3}{4}$

(C) $-\dfrac{4}{3}$

(D) $\dfrac{4}{3}$

Lời giải chi tiết

Differentiating implicitly: $2x + 2y\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{x}{y}$. At $(3, 4)$: $\dfrac{dy}{dx} = -\dfrac{3}{4}$. **(A)**.

Bài tập luyện tập

Find all points on the curve $y^2 = x^3 - 3x + 4$ where the tangent line is horizontal.

Lời giải chi tiết

Differentiating implicitly: $2y\dfrac{dy}{dx} = 3x^2 - 3 \Rightarrow \dfrac{dy}{dx} = \dfrac{3x^2 - 3}{2y}$. A horizontal tangent requires the numerator to vanish (with $y \neq 0$): $3x^2 - 3 = 0 \Rightarrow x = \pm 1$. At $x = 1$: $y^2 = 1 - 3 + 4 = 2 \Rightarrow y = \pm\sqrt{2}$. At $x = -1$: $y^2 = -1 + 3 + 4 = 6 \Rightarrow y = \pm\sqrt{6}$. All four y-values are nonzero, so all are valid. Horizontal tangents occur at $(1, \sqrt{2})$, $(1, -\sqrt{2})$, $(-1, \sqrt{6})$, $(-1, -\sqrt{6})$.

Bài tập luyện tập

The curve $(x - 4)^2 + (y + 3)^2 = 25$ (a circle) has vertical tangent lines at:

(A) $(9, -3)$ and $(-1, -3)$

(B) $(4, 2)$ and $(4, -8)$

(C) $(9, -3)$ only

(D) $(4, 2)$ only

Lời giải chi tiết

Writing the circle as $x^2 + y^2 - 8x + 6y = 0$ and differentiating implicitly: $2x + 2y\dfrac{dy}{dx} - 8 + 6\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{4 - x}{y + 3}$. Vertical tangents require the denominator to vanish: $y = -3$. Substituting into the circle equation: $(x - 4)^2 + 0 = 25 \Rightarrow x = 9$ or $x = -1$. Both give a nonzero numerator ($4 - 9 = -5$ and $4 - (-1) = 5$), so both are valid: $(9, -3)$ and $(-1, -3)$. **(A)**.

Bài tập luyện tập

For the curve $x^2 - xy + y^2 = 3$, find $\dfrac{dy}{dx}$ at the point $(\sqrt{3}, 0)$, and state whether the curve is rising or falling there.

Lời giải chi tiết

First confirm $(\sqrt{3}, 0)$ lies on the curve: $(\sqrt{3})^2 - (\sqrt{3})(0) + 0^2 = 3$, which checks out. From the earlier derivation, $\dfrac{dy}{dx} = \dfrac{y - 2x}{2y - x}$. At $(\sqrt{3}, 0)$: $\dfrac{dy}{dx} = \dfrac{0 - 2\sqrt{3}}{0 - \sqrt{3}} = \dfrac{-2\sqrt{3}}{-\sqrt{3}} = 2$. Since the slope is positive, the curve is **rising** (increasing) as it passes through $(\sqrt{3}, 0)$.

Bài tập luyện tập

Which of the following is *not* part of the definition of a critical point of f at $x = c$?

(A) c is in the domain of f

(B) $f'(c) = 0$ or $f'(c)$ does not exist

(C) f has a local maximum or minimum at $x = c$

(D) All of the above are required

Lời giải chi tiết

A critical point requires only that c be in the domain of f and that $f'(c) = 0$ or fail to exist — it does **not** require c to be a local extremum (many critical points are not extrema, as seen throughout this chapter). **(C)**.

Bài tập luyện tập

Let $f(x) = 3x^5 - 5x^3$. Find all intervals of increase and decrease, classify all critical points with the First Derivative Test, find all intervals of concavity, and find all points of inflection (exact form).

Lời giải chi tiết

Monotonicity: $f'(x) = 15x^4 - 15x^2 = 15x^2(x^2 - 1) = 15x^2(x - 1)(x + 1)$, critical points $x = -1, 0, 1$. Since $x^2 \geq 0$ always, the sign of f' matches the sign of $(x^2 - 1)$: testing $x = -2$ gives $f' > 0$ (increasing); testing $x = -0.5$ and $x = 0.5$ both give $f' < 0$ (decreasing on all of $(-1, 1)$, including straight through $x = 0$); testing $x = 2$ gives $f' > 0$ (increasing). So f increases on $(-\infty, -1)$, decreases on $(-1, 1)$, increases on $(1, \infty)$.

Extrema: at $x = -1$, f' changes $+ \to -$: **local maximum**, $f(-1) = 3(-1) - 5(-1) = 2$. At $x = 1$, f' changes $- \to +$: **local minimum**, $f(1) = 3 - 5 = -2$. At $x = 0$, f' is negative on both sides (no sign change): **not** an extremum.

Concavity: $f''(x) = 60x^3 - 30x = 30x(2x^2 - 1)$, zero at $x = 0$ and $x = \pm\dfrac{1}{\sqrt{2}} = \pm\dfrac{\sqrt{2}}{2}$. Testing: $f''(-1) = -30 < 0$ (down); $f''(-0.5) = 7.5 > 0$ (up); $f''(0.5) = -7.5 < 0$ (down); $f''(1) = 30 > 0$ (up). So f is concave down on $\left(-\infty, -\frac{\sqrt{2}}{2}\right)$, concave up on $\left(-\frac{\sqrt{2}}{2}, 0\right)$, concave down on $\left(0, \frac{\sqrt{2}}{2}\right)$, concave up on $\left(\frac{\sqrt{2}}{2}, \infty\right)$.

Inflection points: sign changes occur at all three roots. $f(0) = 0$. Using f odd (only odd powers of x), $f\left(\frac{\sqrt{2}}{2}\right) = 3\left(\frac{\sqrt{2}}{2}\right)^5 - 5\left(\frac{\sqrt{2}}{2}\right)^3 = 3 \cdot \frac{\sqrt{2}}{8} - 5 \cdot \frac{\sqrt{2}}{4} = \frac{3\sqrt{2}}{8} - \frac{10\sqrt{2}}{8} = -\frac{7\sqrt{2}}{8}$, and by oddness

$f\left(-\frac{\sqrt{2}}{2}\right) = \frac{7\sqrt{2}}{8}$. Inflection points: $\left(-\frac{\sqrt{2}}{2}, \frac{7\sqrt{2}}{8}\right)$, $(0,0)$, $\left(\frac{\sqrt{2}}{2}, -\frac{7\sqrt{2}}{8}\right)$.

Bài tập luyện tập

The function $f(x) = \ln x - x$ (for $x > 0$) has a local maximum at $x =$

(A) 1 **(C)** -1

(B) e **(D)** 0

Lời giải chi tiết

$f'(x) = \frac{1}{x} - 1$, zero when $\frac{1}{x} = 1$, i.e. $x = 1$ (in the domain $x > 0$). $f''(x) = -\frac{1}{x^2}$, so $f''(1) = -1 < 0$, confirming a local maximum by the Second Derivative Test, with value $f(1) = \ln 1 - 1 = -1$. **(A)**.

Bài tập luyện tập

An open-top cylindrical container (a base but no lid) must hold a volume of 27π cm^3. Find the radius and height that minimize the total surface area (base plus lateral surface only), and find that minimum area.

Lời giải chi tiết

Let r be the radius and h the height. Volume: $\pi r^2 h = 27\pi \Rightarrow h = \frac{27}{r^2}$. Surface area (one circular base, no top, plus the lateral surface):

$$S(r) = \pi r^2 + 2\pi r h = \pi r^2 + 2\pi r \cdot \frac{27}{r^2} = \pi r^2 + \frac{54\pi}{r}, \qquad r > 0.$$

$S'(r) = 2\pi r - \frac{54\pi}{r^2}$. Setting $S'(r) = 0$: $2\pi r^3 = 54\pi \Rightarrow r^3 = 27 \Rightarrow r = 3$ cm. Then $h = \frac{27}{9} = 3$ cm. Since $S''(r) = 2\pi + \frac{108\pi}{r^3} > 0$ for every $r > 0$, this critical point is an absolute minimum. Minimum surface area:

$$S(3) = \pi(9) + \frac{54\pi}{3} = 9\pi + 18\pi = 27\pi \text{ cm}^2 \approx 84.8 \text{ cm}^2.$$

Bài tập luyện tập

The curve $xy - x^2 = 4$ passes through the point $(1, 5)$. Find $\frac{dy}{dx}$ at that point, and determine whether the curve is increasing or decreasing there.

Lời giải chi tiết

Check the point: $(1)(5) - (1)^2 = 5 - 1 = 4$, so the point is valid. Differentiate implicitly (product rule on xy):

$$y + x\frac{dy}{dx} - 2x = 0 \implies x\frac{dy}{dx} = 2x - y \implies \frac{dy}{dx} = \frac{2x - y}{x}.$$

At $(1,5)$: $\frac{dy}{dx} = \frac{2(1) - 5}{1} = -3$. Since the slope is negative, the curve is **decreasing** as it passes

through $(1, 5)$.

Ôn tập nhanh: (1) mọi cực trị là điểm tới hạn, nhưng **không** phải mọi điểm tới hạn là cực trị — luôn kiểm tra đổi dấu; (2) Candidates Test = điểm tới hạn **cộng** hai đầu mút, không được thiếu bên nào; (3) $f''(c) = 0$ không tự động là điểm uốn và không làm Second Derivative Test vô hiệu hoàn toàn — nó chỉ buộc quay lại First Derivative Test; (4) bài toán tối ưu luôn cần bước **chứng minh** max/min, không chỉ tìm điểm tới hạn; (5) tiếp tuyến ngang $\Leftrightarrow$ tử số của $\frac{dy}{dx}$ bằng 0; tiếp tuyến đứng $\Leftrightarrow$ mẫu số bằng 0.

Integration and Accumulation of Change

Mục tiêu Unit: Tích lũy sự thay đổi qua diện tích dưới đồ thị hàm tốc độ thay đổi; xấp xỉ diện tích bằng tổng Riemann (trái, phải, trung điểm) và quy tắc hình thang; định nghĩa hình thức của tích phân xác định như giới hạn của tổng Riemann; các tính chất của tích phân xác định (tuyến tính, cộng tính, đổi cận, so sánh); hai phần của Định lý cơ bản của Giải tích (FTC); nguyên hàm và tích phân bất định cùng hằng số $+C$; kỹ thuật đổi biến u cho cả tích phân bất định lẫn xác định; và các phép biến đổi đại số (chia đa thức, tách phân thức, viết lại căn thức) trước khi lấy nguyên hàm.

6.1 Accumulation of Change

Every derivative answers the question "how fast is a quantity changing?" Integration answers the reverse question: given how fast a quantity is changing, how much has it changed in total? If $r(t)$ is the **rate of change** of some quantity Q at time t (with units of "quantity per unit time"), then the **total accumulated change** in Q from $t = a$ to $t = b$ is recovered by adding up all the small contributions $r(t)\,\Delta t$ over the interval — and geometrically, that sum is exactly the **area between the graph of r and the horizontal axis** on $[a, b]$.

The simplest case to picture is motion along a line. If $v(t)$ is the velocity of a particle (its rate of change of position), then over a short time interval Δt the particle's displacement is approximately $v(t)\,\Delta t$ — velocity times elapsed time, which is precisely the area of a thin rectangle of height $v(t)$ and width Δt. Adding up all such thin rectangles across $[a, b]$ and refining the partition gives, in the limit, the exact displacement: the **area under the velocity graph equals displacement**. The same idea applies to any rate: the area under an income-rate graph is total income earned; the area under a flow-rate graph is total volume that has flowed; the area under a population growth-rate graph is the total change in population.

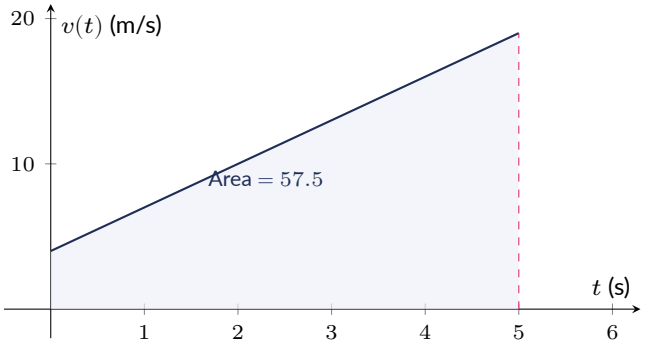
Rate graph → accumulated change

If $r(t)$ is continuous on $[a, b]$, the (signed) area between $y = r(t)$ and the t-axis on $[a, b]$ equals the total accumulated change of the quantity whose rate of change is r. Area above the axis counts as positive change; area below the axis counts as negative change. Units check: (rate) $\times$ (time) $=$ (quantity) — the units of the horizontal axis cancel against the "per unit time" in the vertical axis, leaving the units of the accumulated quantity itself.

VN **GIẢI THÍCH TIẾNG VIỆT**

Đạo hàm trả lời câu hỏi "đại lượng đang thay đổi nhanh thế nào?"; tích phân trả lời câu hỏi ngược lại: "biết tốc độ thay đổi, tổng lượng thay đổi là bao nhiêu?" Nếu $r(t)$ là tốc độ thay đổi của một đại lượng Q, thì **diện tích giữa đồ thị $r(t)$ và trục hoành** trên $[a, b]$ chính là **tổng lượng thay đổi tích lũy** của Q từ $t = a$ đến $t = b$. Trường hợp quen thuộc nhất: diện tích dưới đồ thị vận tốc $v(t)$ bằng độ dời (displacement) của vật. Phần diện tích nằm *trên* trục hoành được tính là thay đổi dương, phần nằm *dưới* trục hoành được tính là thay đổi âm.

When the rate function is linear (or piecewise linear), the "area under the graph" can be computed directly with elementary geometry — triangles, rectangles, and trapezoids — before any integral machinery is introduced. This is the most direct way to see accumulation in action.



Displacement equals the area of the shaded trapezoidal region under $v(t) = 3t + 4$ on $[0, 5]$: a straightforward geometry computation before any integration technique is needed.

Ví dụ mẫu · Displacement as area under a velocity graph

A car travels along a straight road with velocity $v(t) = 3t + 4$ (in m/s) for $0 \le t \le 5$ (in seconds). Find the car's displacement over this interval using geometry alone.

Because $v(t)$ is linear, its graph on $[0, 5]$ is a straight line from $(0, v(0)) = (0, 4)$ to $(5, v(5)) = (5, 19)$, and $v(t) > 0$ throughout, so the region under the graph is a trapezoid with parallel sides of length 4 and 19 and width 5. Its area is

$$\text{Area} = \frac{1}{2}(b_1 + b_2)h = \frac{1}{2}(4 + 19)(5) = \frac{1}{2}(23)(5) = 57.5.$$

Since velocity is positive on the whole interval, the displacement equals this area exactly: the car travels 57.5 **meters** in the forward direction over the first 5 seconds.

For a curved (nonlinear) rate function, no elementary shape describes the exact region, and this is precisely the gap that Riemann sums and the definite integral are built to close: first approximate the area with shapes whose areas we *can* compute (rectangles, trapezoids), and then take these approximations to their limit for an exact value.

6.2 Approximating Areas with Riemann Sums

6.2.1 Left, right, and midpoint sums

To approximate the area under $y = f(x)$ on $[a, b]$, partition the interval into n subintervals of equal width $\Delta x = \dfrac{b - a}{n}$, with partition points $x_0 = a,\ x_1,\ x_2,\ \dots,\ x_n = b$. On each subinterval $[x_{i-1}, x_i]$, build a rectangle of width Δx whose height is the value of f at some chosen point in that subinterval, and add up the n rectangle areas. The three standard choices of "which point" give the **left**, **right**, and **midpoint Riemann sums**.

For n subintervals of equal width $\Delta x = \dfrac{b - a}{n}$:

$$L_n = \Delta x[f(x_0) + f(x_1) + \cdots + f(x_{n-1})], \qquad R_n = \Delta x[f(x_1) + f(x_2) + \cdots + f(x_n)],$$

$$M_n = \Delta x[f(m_1) + f(m_2) + \cdots + f(m_n)],$$

where $m_i = \frac{1}{2}(x_{i-1} + x_i)$ is the midpoint of the i-th subinterval. Each sum is a total area of n rectangles: L_n uses the **left** endpoint of each subinterval as the rectangle's height, R_n uses the **right** endpoint, and M_n uses the **midpoint**.

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Để xấp xỉ diện tích dưới đồ thị $f(x)$ trên $[a,b]$, chia đoạn thành n phần bằng nhau, mỗi phần rộng $\Delta x = \dfrac{b-a}{n}$, rồi dựng n hình chữ nhật có chiều rộng Δx. **Tổng Riemann trái (L_n)** lấy chiều cao mỗi hình chữ nhật bằng giá trị hàm số tại *đầu trái* của mỗi đoạn con; **tổng Riemann phải (R_n)** lấy chiều cao tại *đầu phải*; **tổng Riemann trung điểm (M_n)** lấy chiều cao tại *trung điểm* của mỗi đoạn con. Tổng diện tích các hình chữ nhật là giá trị xấp xỉ cho tích phân xác định.

Whether a given sum over- or under-estimates the true area depends on the shape of f on each subinterval — specifically, on whether f is increasing or decreasing there.

On an interval where f is increasing: the left endpoint gives the smallest function value on each subinterval (rectangle lies below the curve) and the right endpoint gives the largest (rectangle lies above the curve). So L_n **underestimates** and R_n **overestimates** the true area.

On an interval where f is decreasing: the roles reverse — the left endpoint gives the largest value and the right endpoint the smallest. So L_n **overestimates** and R_n **underestimates** the true area.

Ví dụ mẫu · Left, right, and midpoint sums from a formula

Let $f(x) = x^2 + 1$. Approximate $\displaystyle\int_0^4 f(x)\,dx$ using $n = 4$ subintervals of equal width with (a) a left Riemann sum, (b) a right Riemann sum, (c) a midpoint Riemann sum. Then compare all three to the exact value.

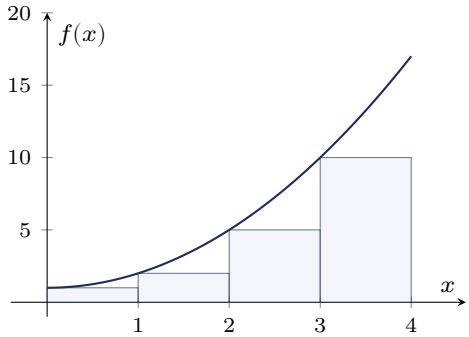
With $n = 4$ on $[0,4]$, $\Delta x = 1$, and the partition points are $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4$, giving

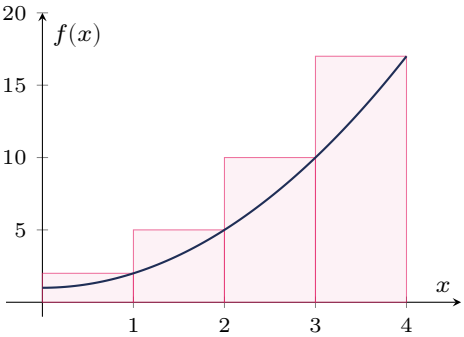
$$f(0) = 1, \quad f(1) = 2, \quad f(2) = 5, \quad f(3) = 10, \quad f(4) = 17.$$

(a) Left sum: $L_4 = \Delta x\big[f(0) + f(1) + f(2) + f(3)\big] = 1(1 + 2 + 5 + 10) = \mathbf{18}$.
(b) Right sum: $R_4 = \Delta x\big[f(1) + f(2) + f(3) + f(4)\big] = 1(2 + 5 + 10 + 17) = \mathbf{34}$.
(c) Midpoint sum: the midpoints are $0.5, 1.5, 2.5, 3.5$, with $f(0.5) = 1.25$, $f(1.5) = 3.25$, $f(2.5) = 7.25$, $f(3.5) = 13.25$, so

$$M_4 = 1(1.25 + 3.25 + 7.25 + 13.25) = \mathbf{25}.$$

The exact value (computed later by antiderivatives) is $\displaystyle\int_0^4 (x^2+1)\,dx = \left[\tfrac{x^3}{3} + x\right]_0^4 = \tfrac{64}{3} + 4 = \tfrac{76}{3} \approx 25.33$. Because f is increasing on $[0,4]$, $L_4 = 18$ underestimates and $R_4 = 34$ overestimates, exactly as predicted; the midpoint sum $M_4 = 25$ lands much closer to the true value than either one-sided sum.





Left sum (left panel, heights at left endpoints) versus right sum (right panel, heights at right endpoints) for $f(x) = x^2 + 1$ on $[0, 4]$ with $n = 4$. Because f is increasing, every left rectangle sits below the curve ($L_4 = 18$, an underestimate) while every right rectangle sits above it ($R_4 = 34$, an overestimate).

(!) Lỗi thường gặp: Do not memorize "left sum = underestimate, right sum = overestimate" as a universal rule — it only holds while f is **increasing**. On a subinterval where f is **decreasing**, the relationship flips: the left endpoint gives the larger value (overestimate) and the right endpoint gives the smaller value (underestimate). Always check monotonicity on the specific interval in question before declaring which sum is too big or too small; on an interval that is increasing in part and decreasing in part, neither one-sided sum needs to be uniformly above or below the true area.

6.2.2 Riemann sums from a table of values

On the AP exam, Riemann sums are frequently built not from a formula but from a **table of values** recorded at discrete times or positions — for instance, sensor readings, flow-rate measurements, or tabulated experimental data. The rectangle-building process is identical; only the source of the function values changes.

Ví dụ mẫu · Left and right sums from a table

Water drains from a tank at rate $R(t)$ (liters per minute), recorded every 5 minutes:

t (min)	0	5	10	15	20
$R(t)$ (L/min)	8	6	5	3	2

Use these five values to estimate $\int_0^{20} R(t)\,dt$, the total volume drained, with (a) a left Riemann sum and (b) a right Riemann sum using $n = 4$ subintervals of width $\Delta t = 5$. State whether each is an over- or underestimate.

(a) $L_4 = 5\big[R(0) + R(5) + R(10) + R(15)\big] = 5(8 + 6 + 5 + 3) = 5(22) =$ **110** liters.
(b) $R_4 = 5\big[R(5) + R(10) + R(15) + R(20)\big] = 5(6 + 5 + 3 + 2) = 5(16) =$ **80** liters.
The data is **decreasing** throughout, so the left endpoint of each subinterval gives the larger reading (rectangle above the true drain-rate curve) while the right endpoint gives the smaller reading (rectangle below the curve): $L_4 = 110$ **overestimates** the total volume drained, and $R_4 = 80$ **underestimates** it.

Ví dụ mẫu · Midpoint sum from a table

Values of a rate function g are recorded at $x = 2, 6, 10, 14$ — chosen so that each is the *midpoint* of one of four subintervals of width 4 partitioning $[0, 16]$:

x (midpoint)	2	6	10	14
$g(x)$	4	9	13	16

Use these values to estimate $\int_0^{16} g(x)\,dx$ with a midpoint Riemann sum.

Each subinterval $[0, 4], [4, 8], [8, 12], [12, 16]$ has width $\Delta x = 4$, and its midpoint value is already given directly in the table, so

$$M_4 = 4\big[g(2) + g(6) + g(10) + g(14)\big] = 4(4 + 9 + 13 + 16) = 4(42) = \textbf{168}.$$

This is the standard table-based setup for a midpoint sum: the given x-values *are* the midpoints of equal-width subintervals spanning the full interval, so no additional averaging of table entries is needed.

6.3 The Trapezoidal Rule

Rather than approximating each strip of area with a flat-topped rectangle, the **Trapezoidal Rule** approximates each strip with a **trapezoid** whose slanted top connects the function values at the two endpoints of the subinterval. This single change usually produces a noticeably better estimate than either one-sided Riemann sum, because a trapezoid follows the general upward or downward trend of the curve across the subinterval instead of using a single, possibly unrepresentative, height.

For n subintervals of equal width $\Delta x = \dfrac{b-a}{n}$ with partition points $x_0, x_1, \dots, x_n$,

$$T_n = \frac{\Delta x}{2}\Big[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\Big].$$

Geometrically, T_n is the sum of the areas of n trapezoids, each with parallel sides $f(x_{i-1})$ and $f(x_i)$ and width Δx. Every interior data point is used **twice** (once as the right edge of one trapezoid and once as the left edge of the next), which is exactly why it carries a coefficient of 2, while the two endpoints $f(x_0)$ and $f(x_n)$ are each used only once.

For equal-width subintervals, the Trapezoidal Rule has a clean relationship to the left and right Riemann sums built from the *same* partition: since each trapezoid's area is the average of the left-rectangle and right-rectangle areas on that subinterval, summing over all subintervals gives $T_n = \frac{1}{2}(L_n + R_n)$ — the Trapezoidal Rule estimate is always exactly the average of the corresponding left and right sums.

Ví dụ mẫu · Trapezoidal Rule from a table, compared to Riemann sums

Using the same drainage-rate table as before —

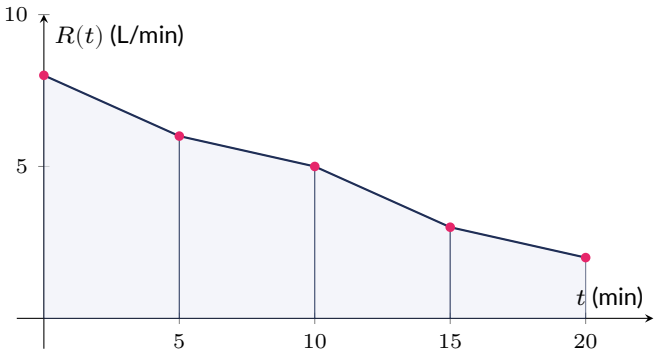
t (min)	0	5	10	15	20
$R(t)$ (L/min)	8	6	5	3	2

estimate $\displaystyle\int_0^{20} R(t)\,dt$ with the Trapezoidal Rule using $n = 4$, and compare to the left and right sums $L_4 = 110$ and $R_4 = 80$ computed earlier.

With $\Delta t = 5$,

$$T_4 = \frac{5}{2}\Big[R(0)+2R(5)+2R(10)+2R(15)+R(20)\Big] = \frac{5}{2}[8+2(6)+2(5)+2(3)+2] = \frac{5}{2}(38) = \textbf{95}\text{ liters.}$$

As expected, $T_4 = 95$ is exactly the average of the left and right sums: $\frac{1}{2}(L_4 + R_4) = \frac{1}{2}(110 + 80) = 95$. It also lies squarely between the overestimate $L_4 = 110$ and the underestimate $R_4 = 80$, illustrating why the trapezoidal estimate is generally more reliable than either one-sided Riemann sum alone.



The Trapezoidal Rule joins consecutive data points with straight segments, forming four trapezoids whose combined area is $T_4 = 95$ liters.

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Quy tắc hình thang (Trapezoidal Rule) thay hình chữ nhật bằng hình thang: cạnh trên nối trực tiếp hai giá trị $f(x_{i-1})$ và $f(x_i)$ ở hai đầu mỗi đoạn con, thay vì lấy một chiều cao cố định. Công thức $T_n = \dfrac{\Delta x}{2}[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$ có hệ số 2 ở mọi điểm *bên trong* vì điểm đó vừa là cạnh phải của hình thang này vừa là cạnh trái của hình thang kế tiếp. Với các đoạn con rộng bằng nhau, T_n luôn đúng bằng **trung bình cộng** của tổng Riemann trái và phải: $T_n = \frac{1}{2}(L_n + R_n)$.

The Trapezoidal Rule's over/underestimate behavior depends on **concavity**, not monotonicity: if f is **concave up** ($f'' > 0$) on $[a, b]$, the straight trapezoid tops lie *above* the curve, so T_n **overestimates** the true integral; if f is **concave down** ($f'' < 0$), the trapezoid tops lie *below* the curve, so T_n **underestimates** it.

6.4 Formal Riemann Sum Notation and the Definite Integral

Summation (Σ) notation lets any of the sums above be written compactly. With $\Delta x = \dfrac{b-a}{n}$ and x_i^* denoting whichever sample point is chosen in the i-th subinterval (left endpoint, right endpoint, midpoint, or any other point in $[x_{i-1}, x_i]$),

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

represents the total area of the n approximating rectangles. As n increases, Δx shrinks, the rectangles hug the curve more and more tightly, and the approximation improves — regardless of which sample point within each subinterval was chosen. This motivates the formal definition of the definite integral as a **limit** of Riemann sums.

Definition of the definite integral. If f is continuous on $[a,b]$, the **definite integral** of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\,\Delta x, \qquad \Delta x = \frac{b-a}{n},$$

provided this limit exists and is the same regardless of which sample points x_i^* are chosen. The symbol $\int$ is an elongated "S" for "sum," $f(x)\,dx$ recalls the "height times infinitesimal width" of each rectangle, and a, b are called the **limits of integration**. Geometrically, $\int_a^b f(x)\,dx$ is the **signed area** between the curve $y=f(x)$ and the x-axis on $[a,b]$: positive where $f\ge 0$, negative where $f\le 0$.

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Tích phân xác định được định nghĩa chính thức là **giới hạn của tổng Riemann** khi số đoạn con $n\to\infty$ (và do đó $\Delta x\to 0$): $\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\,\Delta x$. Miễn là f liên tục trên $[a,b]$, giới hạn này tồn tại và cho cùng một giá trị dù ta chọn điểm mẫu x_i^* ở đầu trái, đầu phải, trung điểm, hay bất kỳ điểm nào trong mỗi đoạn con — đây chính là lý do L_n, R_n, M_n đều hội tụ về *cùng một* giá trị chính xác khi n đủ lớn. Về mặt hình học, tích phân xác định là **diện tích có dấu**: dương khi đồ thị nằm trên trục hoành, âm khi nằm dưới.

Ví dụ mẫu · Evaluating a definite integral from the limit definition

Use the formal limit definition (with right endpoints) to evaluate $\int_0^2 x^2\,dx$.

Partition $[0,2]$ into n equal subintervals: $\Delta x = \dfrac{2}{n}$, and the right endpoint of the i-th subinterval is $x_i = \dfrac{2i}{n}$. Then $f(x_i) = x_i^2 = \dfrac{4i^2}{n^2}$, so

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \frac{4i^2}{n^2}\cdot\frac{2}{n} = \frac{8}{n^3}\sum_{i=1}^{n} i^2.$$

Using the summation formula $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$,

$$R_n = \frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{4(n+1)(2n+1)}{3n^2} = \frac{8n^2+12n+4}{3n^2} = \frac{8}{3}+\frac{4}{n}+\frac{4}{3n^2}.$$

Taking $n\to\infty$, the last two terms vanish, leaving $\int_0^2 x^2\,dx = \lim_{n\to\infty} R_n = \dfrac{8}{3}$. (This matches the antiderivative computation $\left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$ from later in this unit — confirming that the limit-of-Riemann-sums definition and the Fundamental Theorem of Calculus agree.)

6.5 Properties of Definite Integrals

Because the definite integral is built from sums, it inherits several algebraic properties directly from the properties of summation and limits. These properties let complicated integrals be broken apart, combined, or reasoned about without ever computing an antiderivative.

Properties of the definite integral

For continuous functions f, g on the relevant intervals and constants a, b, c, k_1, k_2:

- **Linearity:** $\displaystyle\int_a^b [k_1 f(x) + k_2 g(x)]\,dx = k_1 \int_a^b f(x)\,dx + k_2 \int_a^b g(x)\,dx.$
- **Additivity over adjacent intervals:** $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$ (for a, b, c in any order, as long as f is continuous on the interval containing all three).
- **Reversing the limits of integration:** $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$
- **Zero-width interval:** $\displaystyle\int_a^a f(x)\,dx = 0.$
- **Comparison:** if $f(x) \le g(x)$ for all x in $[a, b]$, then $\displaystyle\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$; and if $m \le f(x) \le M$ on $[a, b]$, then $m(b-a) \le \displaystyle\int_a^b f(x)\,dx \le M(b-a)$.

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Tích phân xác định có các tính chất đại số quan trọng: **tuyến tính** (tách tổng, đưa hằng số ra ngoài); **cộng tính** theo các đoạn kề nhau ($\int_a^b + \int_b^c = \int_a^c$); **đổi cận** thì đổi dấu ($\int_a^b = -\int_b^a$); tích phân trên đoạn có độ dài 0 luôn bằng 0; và tính chất **so sánh**: nếu $f(x) \le g(x)$ trên $[a, b]$ thì $\int_a^b f \le \int_a^b g$, còn nếu $m \le f(x) \le M$ thì tích phân bị "kẹp" giữa $m(b-a)$ và $M(b-a)$. Những tính chất này cho phép tính toán mà không cần tìm nguyên hàm.

Ví dụ mẫu · Combining integrals using properties

Suppose $\displaystyle\int_0^5 f(x)\,dx = 12$, $\displaystyle\int_0^5 g(x)\,dx = 7$, and $\displaystyle\int_5^8 f(x)\,dx = 4$. Find (a) $\displaystyle\int_0^5 [3f(x) - 2g(x)]\,dx$, (b) $\displaystyle\int_0^8 f(x)\,dx$, (c) $\displaystyle\int_5^0 f(x)\,dx$.

(a) By linearity: $3\displaystyle\int_0^5 f(x)\,dx - 2\int_0^5 g(x)\,dx = 3(12) - 2(7) = 36 - 14 = \mathbf{22}$.

(b) By additivity over adjacent intervals: $\displaystyle\int_0^8 f(x)\,dx = \int_0^5 f(x)\,dx + \int_5^8 f(x)\,dx = 12 + 4 = \mathbf{16}$.

(c) By reversing the limits: $\displaystyle\int_5^0 f(x)\,dx = -\int_0^5 f(x)\,dx = \mathbf{-12}$.

6.6 The Fundamental Theorem of Calculus, Part 1

The Fundamental Theorem of Calculus is the bridge that connects the two central operations of calculus — differentiation and integration — showing that they are, in a precise sense, inverse processes. **Part 1** of the theorem addresses what happens when the definite integral's *upper limit* is allowed to vary: define an **accumulation function**

$$F(x) = \int_a^x f(t)\,dt,$$

which reports the accumulated (signed) area under f from the fixed point a up to the variable point x. FTC Part 1 says that differentiating this accumulation function simply returns the original integrand.

FTC Part 1. If f is continuous on an interval containing a, then for every x in that interval,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Chain-rule extension. If the upper limit is itself a differentiable function $g(x)$ instead of x alone,

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x))\cdot g'(x).$$

The extended version is nothing more than FTC Part 1 combined with the chain rule: think of $F(u)=\int_a^u f(t)\,dt$ as the "outer function" (with $F'(u)=f(u)$ by Part 1) and $u=g(x)$ as the "inner function"; then $\frac{d}{dx}F(g(x)) = F'(g(x))\cdot g'(x) = f(g(x))\cdot g'(x)$.

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FTC Phần 1 khẳng định: nếu $F(x)=\int_a^x f(t)\,dt$ (hàm tích lũy, với cận trên là biến x), thì $F'(x)=f(x)$ — lấy đạo hàm "huỷ" tích phân, đưa ta trở lại đúng hàm ban đầu. Khi cận trên không phải là x mà là một hàm $g(x)$, ta áp dụng thêm **quy tắc dây chuyền**: $\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x))\cdot g'(x)$ — lấy f tại cận trên rồi nhân với đạo hàm của cận trên đó.

Ví dụ mẫu · FTC1, direct case

Let $F(x)=\displaystyle\int_2^x (t^3-4t)\,dt$. Find $F'(x)$ and evaluate $F'(3)$.

By FTC Part 1, differentiating simply substitutes x for t in the integrand: $F'(x)=x^3-4x$. Then $F'(3)=3^3-4(3)=27-12=$ **15**.

Ví dụ mẫu · FTC1 with the chain rule and a variable lower limit

(a) Let $G(x)=\displaystyle\int_1^{x^2} \sin(t)\,dt$. Find $G'(x)$.

(b) Let $H(x)=\displaystyle\int_x^5 \sqrt{t^2+1}\,dt$. Find $H'(x)$.

(a) The upper limit is $g(x)=x^2$, with $g'(x)=2x$. By the chain-rule extension of FTC1,

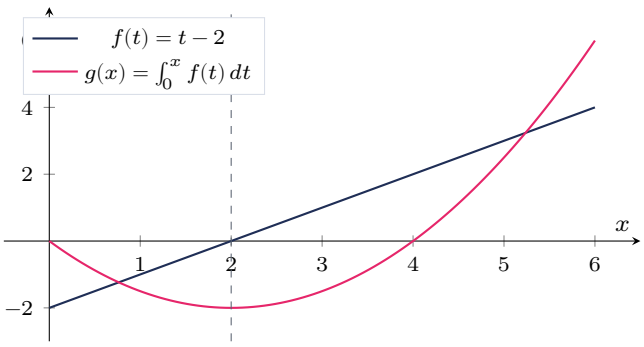
$$G'(x)=\sin(x^2)\cdot\frac{d}{dx}[x^2]=2x\sin(x^2).$$

(b) Here the *variable* is the **lower** limit, not the upper one, so first reverse the limits of integration to move the variable to the top:

$$H(x)=\int_x^5 \sqrt{t^2+1}\,dt=-\int_5^x \sqrt{t^2+1}\,dt.$$

Now FTC Part 1 applies directly to the rewritten form (upper limit x, lower limit fixed at 5):

$$H'(x)=-\sqrt{x^2+1}.$$



The accumulation function $g(x) = \int_0^x (t-2)\,dt = \frac{x^2}{2} - 2x$ alongside $f(t) = t - 2$. Where $f < 0$ (on $[0, 2]$), g is decreasing; where $f > 0$ (on $[2, 6]$), g is increasing; g has a minimum exactly where f crosses zero, at $x = 2$ — a direct visual statement of $g'(x) = f(x)$.

(!) Lỗi thường gặp: When the variable of integration appears in the **lower** limit instead of the upper limit, do *not* apply FTC1 directly and forget the sign. $\dfrac{d}{dx}\displaystyle\int_x^b f(t)\,dt$ is **not** $f(x)$ — it is $-f(x)$, because reversing the limits to put the variable on top, $\int_x^b f(t)\,dt = -\int_b^x f(t)\,dt$, introduces a minus sign before FTC1 can be applied. More generally, for $\dfrac{d}{dx}\int_{h(x)}^{g(x)} f(t)\,dt$, split at any convenient constant c in the domain: $\int_{h(x)}^{c} f(t)\,dt + \int_c^{g(x)} f(t)\,dt = -\int_c^{h(x)} f(t)\,dt + \int_c^{g(x)} f(t)\,dt$, then differentiate each piece separately: $-f(h(x))h'(x) + f(g(x))g'(x)$.

6.7 The Fundamental Theorem of Calculus, Part 2

While Part 1 differentiates an accumulation function, **Part 2** goes the other direction: it turns evaluating a definite integral into a simple two-step arithmetic process, using *any* antiderivative of the integrand — no limits, no Riemann sums required.

FTC Part 2. If f is continuous on $[a, b]$ and F is **any** antiderivative of f (that is, $F' = f$) on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

often written $\big[F(x)\big]_a^b$. Because two antiderivatives of the same function differ only by a constant, the choice of antiderivative (and, in particular, the value of $+C$) does not affect the result: the constant would appear in both $F(b)$ and $F(a)$ and cancel in the subtraction.

VN **GIẢI THÍCH TIẾNG VIỆT**

FTC Phần 2 biến việc tính tích phân xác định thành phép trừ đơn giản: tìm *một* nguyên hàm $F(x)$ bất kỳ của $f(x)$, rồi tính $F(b) - F(a)$. Vì hai nguyên hàm bất kỳ của cùng một hàm chỉ chênh nhau một hằng số C, hằng số này luôn bị triệt tiêu khi lấy hiệu $F(b) - F(a)$ — do đó khi tính tích phân xác định, ta **không cần** viết $+C$.

Ví dụ mẫu · Evaluating a definite integral with FTC2

Evaluate $\displaystyle\int_1^2 \left(3x^2 - \frac{1}{x^2}\right) dx$.

An antiderivative of $3x^2 - x^{-2}$ is $F(x) = x^3 + \dfrac{1}{x}$ (since $\dfrac{d}{dx}[x^{-1}] = -x^{-2}$, the antiderivative of $-x^{-2}$ is $+x^{-1}$). By FTC2,

$$\int_1^2 \left(3x^2 - \frac{1}{x^2}\right) dx = \left[x^3 + \frac{1}{x}\right]_1^2 = \left(2^3 + \frac{1}{2}\right) - (1^3 + 1)$$
$$= \left(8 + \frac{1}{2}\right) - (1 + 1) = \frac{17}{2} - 2 = \frac{17}{2} - \frac{4}{2} = \frac{13}{2}.$$

The value of the integral is $\boxed{\dfrac{13}{2}}$.

Ví dụ mẫu · Two more applications of FTC2

(a) Evaluate $\displaystyle\int_0^{\pi/2} \cos x\, dx$. (b) Evaluate $\displaystyle\int_{-1}^{2} (x^3 - 2x)\, dx$.

(a) An antiderivative of $\cos x$ is $\sin x$:

$$\int_0^{\pi/2} \cos x\, dx = \left[\sin x\right]_0^{\pi/2} = \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1 - 0 = \mathbf{1}.$$

(b) An antiderivative of $x^3 - 2x$ is $\dfrac{x^4}{4} - x^2$:

$$\int_{-1}^{2} (x^3 - 2x)\, dx = \left[\frac{x^4}{4} - x^2\right]_{-1}^{2} = \left(\frac{16}{4} - 4\right) - \left(\frac{1}{4} - 1\right) = (4 - 4) - \left(-\frac{3}{4}\right) = 0 + \frac{3}{4} = \mathbf{\frac{3}{4}}.$$

6.8 Antiderivatives and Indefinite Integrals

An **antiderivative** of f is any function F with $F'(x) = f(x)$. Antiderivatives are never unique: if F is one antiderivative of f, then $F(x) + C$ is also an antiderivative for *every* constant C, since $\dfrac{d}{dx}[F(x) + C] = F'(x) + 0 = f(x)$. Conversely, any two antiderivatives of the same continuous function on an interval differ by exactly a constant — so the collection of *all* antiderivatives of f is the single family $F(x) + C$. This family is written as the **indefinite integral**,

$$\int f(x)\, dx = F(x) + C,$$

where C is called the **constant of integration**. Graphically, the graphs of $F(x)+C$ for different values of C are vertical translates of one another — an entire family of parallel curves, all sharing the same slope $f(x)$ at every x-value, but stacked at different heights.

Basic antiderivative rules.

Function	Antiderivative	Function	Antiderivative
$x^n\ (n \neq -1)$	$\dfrac{x^{n+1}}{n+1} + C$	$\sin x$	$-\cos x + C$
$\dfrac{1}{x}$	$\ln\lvert x\rvert + C$	$\cos x$	$\sin x + C$
e^x	$e^x + C$	$\sec^2 x$	$\tan x + C$
$\csc^2 x$	$-\cot x + C$	$\sec x \tan x$	$\sec x + C$

Each rule is simply the corresponding differentiation rule read backwards; every entry can be checked by differentiating the right-hand column and confirming it returns the left-hand function.

VN **GIẢI THÍCH TIẾNG VIỆT**

Nguyên hàm (antiderivative) của f là bất kỳ hàm F nào thoả $F'(x) = f(x)$. Vì đạo hàm của một hằng số luôn bằng 0, nếu $F(x)$ là một nguyên hàm thì $F(x) + C$ cũng là nguyên hàm với **mọi** hằng số C — do đó **tích phân bất định** $\int f(x)\,dx = F(x) + C$ đại diện cho *cả một họ* đường cong, tất cả có cùng độ dốc $f(x)$ tại mỗi điểm nhưng nằm ở các độ cao khác nhau (các đường cong này chỉ là tịnh tiến theo phương thẳng đứng của nhau). Bảng nguyên hàm cơ bản chính là bảng đạo hàm "đọc ngược".

(!) Lỗi thường gặp: Forgetting the "$+C$" on an **indefinite** integral is one of the most common point losses on the AP exam. $\displaystyle\int f(x)\,dx$ represents an entire family of antiderivatives, not a single function — omitting $+C$ effectively claims there is only one antiderivative, which is false for every function. (The constant *does* disappear when evaluating a **definite** integral by FTC2, because it cancels in the subtraction $F(b) - F(a)$ — but that cancellation is a consequence of having limits of integration, and does not apply to an indefinite integral.)

(!) Lỗi thường gặp: The reverse power rule $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ **fails** at $n = -1$, because the formula would require dividing by $n + 1 = 0$. So $\displaystyle\int \frac{1}{x}\,dx$ is **not** $\dfrac{x^0}{0}$ (undefined) — it is $\ln\lvert x\rvert + C$, a separate rule that must be memorized on its own. The absolute value is essential: $\ln\lvert x\rvert$ is defined for all $x \neq 0$ (both positive and negative), matching the domain of $\dfrac{1}{x}$, whereas $\ln x$ alone would only be defined for $x > 0$.

Ví dụ mẫu · Finding a particular antiderivative and applying the basic rules

(a) Find $f(x)$ given $f'(x) = 4x^3 - 6x + 5$ and $f(1) = 3$.
(b) Evaluate $\displaystyle\int \left(3e^x + \frac{2}{x} - 5\sec^2 x\right) dx$.

(a) Antidifferentiate term by term: $f(x) = x^4 - 3x^2 + 5x + C$. Apply the initial condition $f(1) = 3$:

$$1 - 3 + 5 + C = 3 \implies 3 + C = 3 \implies C = 0.$$

So this initial condition picks out exactly one curve from the family: $f(x) = x^4 - 3x^2 + 5x$.

(b) Applying the table of rules to each term:

$$\int \left(3e^x + \frac{2}{x} - 5\sec^2 x\right) dx = 3e^x + 2\ln|x| - 5\tan x + C.$$

6.9 Integration by u-Substitution

6.9.1 Indefinite integrals

Many integrands are the result of a chain-rule differentiation, and u**-substitution** is essentially the chain rule run in reverse. If an integrand can be seen as $f(g(x)) \cdot g'(x)$ for some "inner function" g, substituting $u = g(x)$ (so $du = g'(x)\,dx$) rewrites the whole integral in terms of the single variable u, often reducing it to a basic rule from the previous section.

u-substitution procedure

1. Identify an inner function $u = g(x)$ whose derivative $g'(x)$ (up to a constant multiple) also appears in the integrand.
2. Compute $du = g'(x)\,dx$ and solve for dx (or for whatever combination, such as $x\,dx$, is needed to match the integrand).
3. Rewrite the entire integral in terms of u only — no x's should remain.
4. Integrate with respect to u using the basic rules, then substitute $g(x)$ back in for u (for an **indefinite** integral only — see the next subsection for definite integrals).

VN **GIẢI THÍCH TIẾNG VIỆT**

Đổi biến u (u-substitution) thực chất là "quy tắc dây chuyền chạy ngược". Chọn u là một "hàm trong" sao cho đạo hàm của nó (sai khác một hằng số) cũng xuất hiện trong biểu thức tích phân; tính du; viết lại toàn bộ tích phân theo u (không còn x); tích phân theo u bằng các quy tắc cơ bản; và với tích phân **bất định**, thay u trở lại bằng $g(x)$ ở bước cuối để đưa đáp án về biến x ban đầu.

Ví dụ mẫu · Indefinite integral by u-substitution

Evaluate $\displaystyle\int x\sqrt{x^2+1}\,dx$.

Let $u = x^2 + 1$, so $du = 2x\,dx$, which gives $x\,dx = \dfrac{du}{2}$. Substituting,

$$\int x\sqrt{x^2+1}\,dx = \int \sqrt{u}\cdot\frac{du}{2} = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C = \frac{1}{3}u^{3/2} + C.$$

Substituting $u = x^2 + 1$ back in (since this is an indefinite integral):

$$\int x\sqrt{x^2+1}\,dx = \frac{1}{3}(x^2+1)^{3/2} + C.$$

Check: $\dfrac{d}{dx}\left[\dfrac{1}{3}(x^2+1)^{3/2}\right] = \dfrac{1}{3}\cdot\dfrac{3}{2}(x^2+1)^{1/2}\cdot 2x = x(x^2+1)^{1/2} = x\sqrt{x^2+1}$, matching the original integrand.

6.9.2 Definite integrals: changing the limits

When u-substitution is applied to a **definite** integral, there are two valid strategies: substitute back to x at the end and evaluate at the original x-limits (as in an indefinite integral), or — more efficiently — **convert the limits of integration to u-values** at the moment of substitution, so the entire remaining computation stays in terms of u and no back-substitution is ever needed.

For $\displaystyle\int_a^b f(g(x))\,g'(x)\,dx$ with $u = g(x)$: once $du = g'(x)\,dx$ is substituted, also replace the limits of integration — the new lower limit is $u = g(a)$ and the new upper limit is $u = g(b)$ — and evaluate the resulting u-integral directly at these new limits:

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

(!) Lỗi thường gặp: The single most common error in definite-integral u-substitution is finding the antiderivative in terms of u and then plugging the **original x-limits** directly into it, without converting them to u-limits and without substituting back to x first. Both of the following are correct: (1) change the limits to $u = g(a)$ and $u = g(b)$ and evaluate the u-antiderivative at those u-values, or (2) substitute back to get the antiderivative in terms of x and evaluate at the original x-limits. Mixing the two — a u-antiderivative evaluated at x-limits — silently produces a wrong numeric answer with no error message to warn you.

Ví dụ mẫu · Definite integral with changed limits

Evaluate $\displaystyle\int_0^1 6x^2(x^3+2)^4\,dx$.

Let $u = x^3 + 2$, so $du = 3x^2\,dx$, which gives $x^2\,dx = \dfrac{du}{3}$. Convert the limits: when $x = 0$, $u = 0^3 + 2 = 2$; when $x = 1$, $u = 1^3 + 2 = 3$. Substituting everything, including the new limits,

$$\begin{aligned}\int_0^1 6x^2(x^3+2)^4\,dx &= \int_2^3 6u^4 \cdot \frac{du}{3} = \int_2^3 2u^4\,du = \left[\frac{2}{5}u^5\right]_2^3 \\ &= \frac{2}{5}(3^5 - 2^5) = \frac{2}{5}(243 - 32) = \frac{2}{5}(211) = \frac{422}{5}.\end{aligned}$$

Notice the final numerical evaluation used only u-limits (2 and 3) — the original x-limits (0 and 1) were never needed again once they were converted.

Ví dụ mẫu · A second definite u-substitution

Evaluate $\displaystyle\int_1^e \frac{(\ln x)^2}{x}\,dx$.

Let $u = \ln x$, so $du = \dfrac{dx}{x}$. Convert the limits: when $x = 1$, $u = \ln 1 = 0$; when $x = e$, $u = \ln e = 1$. Then

$$\int_1^e \frac{(\ln x)^2}{x}\,dx = \int_0^1 u^2\,du = \left[\frac{u^3}{3}\right]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}.$$

6.10 Algebraic Manipulation Before Integrating

Not every integrand is ready to integrate in the form it is given. A short algebraic rewrite — dividing, splitting a fraction into separate terms, or converting a radical to a fractional power — often turns an intractable-looking expression into a sum of pieces that each match a basic rule directly.

Three algebraic rewriting techniques

- **Polynomial long division:** if the integrand is an "improper" rational function (degree of numerator $\geq$ degree of denominator), divide first to rewrite it as *polynomial* $+ \dfrac{\text{remainder}}{\text{divisor}}$, a sum the basic rules (including $\ln|x|$-type terms) can each handle separately.
- **Splitting a fraction:** if the numerator is a sum but the denominator is a single term, split $\dfrac{A+B+C}{D}$ into $\dfrac{A}{D}+\dfrac{B}{D}+\dfrac{C}{D}$ and simplify each piece into a power of x.
- **Rewriting radicals as fractional powers:** replace $\sqrt[n]{x^m}$ with $x^{m/n}$ (and $\dfrac{1}{\sqrt[n]{x^m}}$ with $x^{-m/n}$) so the reverse power rule applies directly.

VN **GIẢI THÍCH TIẾNG VIỆT**

Không phải biểu thức nào cũng sẵn sàng để lấy nguyên hàm ngay — đôi khi cần biến đổi đại số trước: **chia đa thức** khi tử có bậc $\geq$ bậc mẫu (phân thức "không thực sự"), đưa về dạng đa thức cộng với phân thức đơn giản hơn; **tách phân thức** khi mẫu chỉ có một số hạng nhưng tử có nhiều số hạng, tách thành tổng các phân thức luỹ thừa của x; và **viết căn thức dưới dạng luỹ thừa phân số** ($\sqrt[n]{x^m} = x^{m/n}$) để áp dụng trực tiếp quy tắc luỹ thừa ngược.

Ví dụ mẫu · Algebraic manipulation before integrating

(a) Evaluate $\displaystyle\int \frac{x^2+3x+5}{x+1}\,dx$, then use the result to evaluate $\displaystyle\int_0^2 \frac{x^2+3x+5}{x+1}\,dx$.

(b) Evaluate $\displaystyle\int \frac{2x^3-5x+7}{x^2}\,dx$.

(c) Evaluate $\displaystyle\int_1^4 \left(2\sqrt{x}-\frac{3}{\sqrt{x}}\right)dx$.

(a) Long division: x^2+3x+5 divided by $x+1$ gives quotient $x+2$ with remainder 3 (check: $(x+1)(x+2)+3 = x^2+3x+2+3 = x^2+3x+5$), so

$$\frac{x^2+3x+5}{x+1} = x+2+\frac{3}{x+1} \implies \int \frac{x^2+3x+5}{x+1}\,dx = \frac{x^2}{2}+2x+3\ln|x+1|+C.$$

Evaluating from 0 to 2:

$$\left[\frac{x^2}{2}+2x+3\ln|x+1|\right]_0^2 = (2+4+3\ln 3)-(0+0+3\ln 1) = 6+3\ln 3.$$

(b) Split the fraction term by term: $\dfrac{2x^3-5x+7}{x^2} = 2x-\dfrac{5}{x}+7x^{-2}$, so

$$\int \left(2x-\frac{5}{x}+7x^{-2}\right)dx = x^2-5\ln|x|-\frac{7}{x}+C.$$

(c) Rewrite the radicals: $2\sqrt{x} = 2x^{1/2}$ and $\dfrac{3}{\sqrt{x}} = 3x^{-1/2}$, so an antiderivative is

$$F(x) = \frac{4}{3}x^{3/2}-6x^{1/2}+C.$$

Evaluating (with $4^{3/2}=8$, $4^{1/2}=2$, $1^{3/2}=1^{1/2}=1$):

$$F(4)-F(1)=\left(\frac{4}{3}(8)-6(2)\right)-\left(\frac{4}{3}(1)-6(1)\right)=\left(\frac{32}{3}-12\right)-\left(\frac{4}{3}-6\right)=\left(-\frac{4}{3}\right)-\left(-\frac{14}{3}\right)=\frac{10}{3}.$$

6.11 Integration by Parts

Phần dành riêng cho AP Calculus BC (CED 6.11–6.14): tích phân từng phần (kể cả phương pháp bảng và tích phân "lặp vòng"), tách phân thức với mẫu là tích các nhân tử bậc nhất phân biệt, tích phân suy rộng (cận vô hạn và hàm không bị chặn), và chiến lược chọn kỹ thuật lấy nguyên hàm. Các mục này chỉ xuất hiện trong đề BC.

u-substitution undoes the chain rule. **Integration by parts** undoes the **product rule**, and it is the tool for integrands that are a product of two unrelated kinds of function — a polynomial times an exponential, a polynomial times a trigonometric function, a logarithm times a power of x — where no inner function and its derivative appear together.

6.11.1 Deriving the formula

If u and v are differentiable functions of x, the product rule says $\frac{d}{dx}[uv]=u\,\frac{dv}{dx}+v\,\frac{du}{dx}$. Integrating both sides with respect to x gives $uv=\int u\,dv+\int v\,du$, and solving for the first integral produces the formula.

Integration by parts

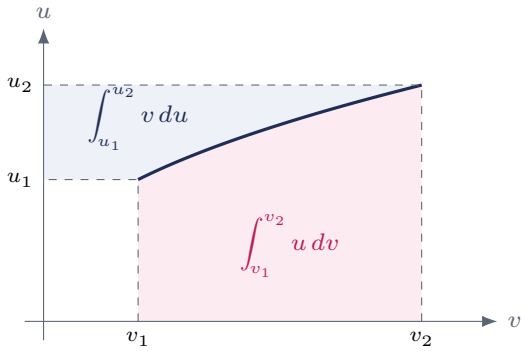
$$\int u\,dv \;=\; uv-\int v\,du \qquad\qquad \int_a^b u\,dv \;=\; \Big[uv\Big]_a^b-\int_a^b v\,du$$

Procedure. (1) Split the integrand into a factor u (to be differentiated) and a factor dv (to be integrated; dv *includes* dx). (2) Compute $du=u'(x)\,dx$ and $v=\int dv$ (any one antiderivative — no $+C$ needed here). (3) Substitute into $uv-\int v\,du$. The method succeeds when the new integral $\int v\,du$ is *simpler* than the original.

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Tích phân từng phần là "quy tắc đạo hàm của tích chạy ngược": $\int u\,dv=uv-\int v\,du$. Ta tách biểu thức dưới dấu tích phân thành hai phần — phần u sẽ được *lấy đạo hàm* (nên chọn phần trở nên đơn giản hơn khi đạo hàm, như đa thức hay $\ln x$) và phần dv sẽ được *lấy nguyên hàm* (phải là phần ta biết lấy nguyên hàm, như e^x, $\sin x$, $\cos x$). Công thức chỉ có ích khi tích phân mới $\int v\,du$ dễ hơn tích phân ban đầu. Với tích phân xác định, phần uv cũng phải được thay cận: $[uv]_a^b$.

The formula has a clean geometric meaning. If u increases from u_1 to u_2 while v increases from v_1 to v_2, the two integrals $\int u\,dv$ and $\int v\,du$ are the two pieces into which the curve splits the "L-shaped" difference of two rectangles:



Pink area + navy area $= u_2v_2 - u_1v_1 = [uv]$, which rearranges to $\int u\,dv = [uv] - \int v\,du$.

6.11.2 Choosing u: the LIATE guideline

The whole method rests on a good choice of u. A reliable rule of thumb is to choose u as the factor that appears *first* in the list below; whatever is left (including dx) becomes dv.

	Type	Examples	Why it makes a good u
L	Logarithmic	$\ln x$, $\ln(2x+1)$	derivative $\frac{1}{x}$ is algebraic; hard to integrate directly
I	Inverse trigonometric	$\arctan x$, $\arcsin x$	derivative is algebraic
A	Algebraic (polynomial)	x, x^2, $3x-1$	each derivative lowers the degree
T	Trigonometric	$\sin x$, $\cos 3x$	cycles under differentiation
E	Exponential	e^x, e^{-2x}	easiest to integrate — best left as dv

Ví dụ mẫu · Polynomial times exponential

Evaluate $\displaystyle\int xe^{2x}\,dx$.

The factors are algebraic (x) and exponential (e^{2x}); A comes before E, so take

$$u = x, \quad dv = e^{2x}\,dx \qquad \Longrightarrow \qquad du = dx, \quad v = \tfrac{1}{2}e^{2x}.$$

Then

$$\int xe^{2x}\,dx = x\cdot\tfrac{1}{2}e^{2x} - \int \tfrac{1}{2}e^{2x}\,dx = \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + C.$$

Check: $\dfrac{d}{dx}\left[\dfrac{x}{2}e^{2x} - \dfrac{1}{4}e^{2x}\right] = \dfrac{1}{2}e^{2x} + xe^{2x} - \dfrac{1}{2}e^{2x} = xe^{2x}$. ✓

Had we chosen $u = e^{2x}$ and $dv = x\,dx$ instead, the new integral would be $\int \frac{x^2}{2}\cdot 2e^{2x}\,dx$ — *worse* than the original, since the power of x went up.

Ví dụ mẫu · Polynomial times trigonometric

Evaluate $\displaystyle\int x\cos x\,dx$.

Take $u = x$, $dv = \cos x\,dx$, so $du = dx$ and $v = \sin x$:

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C.$$

Check: $\dfrac{d}{dx}[x\sin x + \cos x] = \sin x + x\cos x - \sin x = x\cos x$. ✓

6.11.3 Integrals with a "hidden" $dv = dx$

Some single functions — $\ln x$, $\arctan x$, $\arcsin x$ — have no basic antiderivative rule, but they have simple derivatives. Treat them as a product with 1: choose u to be the function itself and $dv = dx$.

Ví dụ mẫu · The integrals of $\ln x$ and $\arctan x$

Evaluate **(a)** $\int \ln x\,dx$ and **(b)** $\int \arctan x\,dx$.

(a) $u = \ln x$, $dv = dx \Rightarrow du = \dfrac{1}{x}\,dx$, $v = x$:

$$\int \ln x\,dx = x\ln x - \int x\cdot\frac{1}{x}\,dx = x\ln x - \int 1\,dx = x\ln x - x + C.$$

(b) $u = \arctan x$, $dv = dx \Rightarrow du = \dfrac{1}{1+x^2}\,dx$, $v = x$:

$$\int \arctan x\,dx = x\arctan x - \int \frac{x}{1+x^2}\,dx.$$

The remaining integral is a u-substitution ($w = 1+x^2$, $dw = 2x\,dx$): $\displaystyle\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\ln(1+x^2) + C$. Hence

$$\int \arctan x\,dx = x\arctan x - \tfrac{1}{2}\ln(1+x^2) + C.$$

(No absolute value is needed because $1+x^2 > 0$.) Both results are worth remembering — they appear inside larger problems.

6.11.4 Repeated integration by parts

If u is a polynomial of degree n, one application lowers the degree by one, so n applications are needed. Keep the same type of choice ($u =$ the polynomial part) each time — switching mid-problem simply undoes the previous step.

Ví dụ mẫu · Two applications

Evaluate $\displaystyle\int x^2e^x\,dx$.

First application: $u = x^2$, $dv = e^x\,dx \Rightarrow du = 2x\,dx$, $v = e^x$:

$$\int x^2e^x\,dx = x^2e^x - \int 2xe^x\,dx.$$

Second application (on $\int 2xe^x\,dx$): $u = 2x$, $dv = e^x\,dx \Rightarrow du = 2\,dx$, $v = e^x$:

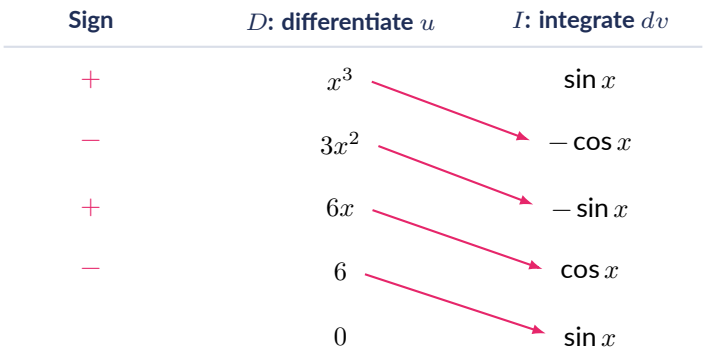
$$\int 2xe^x\,dx = 2xe^x - \int 2e^x\,dx = 2xe^x - 2e^x + C.$$

Combine (watch the sign — the whole second result is subtracted):

$$\int x^2e^x\,dx = x^2e^x - (2xe^x - 2e^x) + C = e^x(x^2 - 2x + 2) + C.$$

6.11.5 The tabular method

When u is a polynomial that eventually differentiates to 0, and dv can be integrated repeatedly without difficulty, the bookkeeping of repeated integration by parts can be organized in a table. Write the successive **derivatives** of u in one column (until reaching 0) and successive **antiderivatives** of dv in the other. Then multiply along the diagonals, attaching alternating signs $+, -, +, -, \dots$, and add.



Tabular method for $\int x^3 \sin x\, dx$: each pink diagonal arrow is one product, taken with the sign in its row. The process stops when the D-column reaches 0.

Ví dụ mẫu · Tabular integration

Evaluate $\displaystyle\int x^3 \sin x\, dx$.

Reading the four diagonals of the table above with signs $+, -, +, -$:

$$\int x^3 \sin x\, dx = (+)\, x^3(-\cos x)\ (-)\ 3x^2(-\sin x)\ (+)\ 6x(\cos x)\ (-)\ 6(\sin x) + C$$

$$= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C.$$

This is exactly what three applications of integration by parts would give, with every sign handled automatically by the alternating column.

VN **GIẢI THÍCH TIẾNG VIỆT**

Phương pháp bảng (tabular) dùng khi u là đa thức (đạo hàm nhiều lần sẽ về 0) và dv lấy nguyên hàm liên tiếp được dễ dàng. Cột D ghi các đạo hàm liên tiếp của u đến khi bằng 0; cột I ghi các nguyên hàm liên tiếp của dv. Nhân theo **đường chéo** (hàng k của cột D với hàng $k+1$ của cột I), gắn dấu luân phiên $+, -, +, -, \dots$ rồi cộng lại. Lỗi hay gặp nhất là quên đổi dấu, hoặc lấy nguyên hàm sai dấu của $\sin x$ và $\cos x$ ($\int \sin x\, dx = -\cos x$).

6.11.6 Integrals that return to themselves

For products such as $e^x \sin x$ or $e^{2x} \cos x$, neither factor ever differentiates to 0. Two applications of integration by parts bring back the *original* integral; it can then be solved for algebraically, like an unknown in an equation.

Ví dụ mẫu · A cyclic integral

Evaluate $I = \displaystyle\int e^x \sin x\, dx$.

First application: $u = \sin x$, $dv = e^x\,dx \Rightarrow du = \cos x\,dx$, $v = e^x$:

$$I = e^x \sin x - \int e^x \cos x\,dx.$$

Second application (keep u trigonometric!): $u = \cos x$, $dv = e^x\,dx \Rightarrow du = -\sin x\,dx$, $v = e^x$:

$$\int e^x \cos x\,dx = e^x \cos x - \int e^x(-\sin x)\,dx = e^x \cos x + I.$$

Substitute back: $I = e^x \sin x - (e^x \cos x + I) = e^x \sin x - e^x \cos x - I$. Adding I to both sides, $2I = e^x(\sin x - \cos x)$, so

$$\int e^x \sin x\,dx = \frac{e^x(\sin x - \cos x)}{2} + C.$$

Check: $\frac{d}{dx}\left[\frac{1}{2}e^x(\sin x - \cos x)\right] = \frac{1}{2}e^x(\sin x - \cos x) + \frac{1}{2}e^x(\cos x + \sin x) = e^x \sin x$. ✓

(!) Lỗi thường gặp: In a cyclic integral, if the second application uses $u = e^x$ (the *opposite* choice from the first), the computation collapses to the useless identity $I = I$. Keep the same type of function as u in both steps. Also, the constant $+C$ is added only at the very end, after solving for I.

6.11.7 Definite integrals by parts

For a definite integral, the boundary term uv must be evaluated at the limits too:

$$\int_a^b u\,dv = \Big[uv\Big]_a^b - \int_a^b v\,du.$$

Ví dụ mẫu · Two definite integrals

Evaluate **(a)** $\displaystyle\int_0^{\pi} x \sin x\,dx$ and **(b)** $\displaystyle\int_1^{e} x \ln x\,dx$.

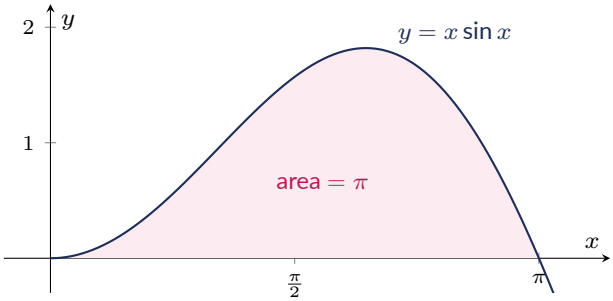
(a) $u = x$, $dv = \sin x\,dx \Rightarrow du = dx$, $v = -\cos x$:

$$\int_0^{\pi} x \sin x\,dx = \Big[-x\cos x\Big]_0^{\pi} + \int_0^{\pi} \cos x\,dx = (-\pi\cos\pi + 0) + \Big[\sin x\Big]_0^{\pi} = \pi + 0 = \pi.$$

(b) L comes before A, so $u = \ln x$, $dv = x\,dx \Rightarrow du = \dfrac{1}{x}\,dx$, $v = \dfrac{x^2}{2}$:

$$\int_1^{e} x \ln x\,dx = \left[\frac{x^2}{2}\ln x\right]_1^{e} - \int_1^{e} \frac{x^2}{2}\cdot\frac{1}{x}\,dx = \frac{e^2}{2} - \left[\frac{x^2}{4}\right]_1^{e} = \frac{e^2}{2} - \frac{e^2-1}{4} = \frac{e^2+1}{4}.$$

(The boundary term at $x = 1$ vanishes because $\ln 1 = 0$.)



The region under one arch of $y = x \sin x$ has area exactly π — a value reachable only through integration by parts.

6.11.8 Integration by parts with tables of values

A favorite free-response format gives a table of values of f and f' (and sometimes f'') and asks for an integral such as $\int x\, f'(x)\, dx$. No formula for f is available, so integration by parts is the only way in: it trades the unknown integral for boundary values from the table plus an integral that is given.

Ví dụ mẫu · FRQ-style table

The function f is twice differentiable, and selected values of f and f' are shown below. It is known that $\displaystyle\int_1^4 f(x)\, dx = 14$.

x	1	2	4
$f(x)$	5	4	7
$f'(x)$	-2	0	6

(a) Evaluate $\displaystyle\int_1^4 x\, f'(x)\, dx$. **(b)** Evaluate $\displaystyle\int_1^2 x\, f''(x)\, dx$.

(a) Let $u = x$, $dv = f'(x)\, dx$, so $du = dx$ and $v = f(x)$:

$$\int_1^4 x\, f'(x)\, dx = \Big[x\, f(x)\Big]_1^4 - \int_1^4 f(x)\, dx = (4 \cdot 7 - 1 \cdot 5) - 14 = 23 - 14 = 9.$$

(b) Let $u = x$, $dv = f''(x)\, dx$, so $du = dx$ and $v = f'(x)$:

$$\int_1^2 x\, f''(x)\, dx = \Big[x\, f'(x)\Big]_1^2 - \int_1^2 f'(x)\, dx = (2 \cdot 0 - 1 \cdot (-2)) - (f(2) - f(1)) = 2 - (4 - 5) = 3.$$

The last integral used the Fundamental Theorem: $\int_1^2 f'(x)\, dx = f(2) - f(1)$.

(!) Lỗi thường gặp: Three classic errors: (1) dropping the boundary term $[uv]_a^b$ in a definite integral, or evaluating it only at the upper limit; (2) losing the minus sign in front of $\int v\, du$ when the result of a second application is substituted back — always wrap it in parentheses; (3) choosing $u = e^x$ or $u = \sin x$ with a polynomial dv, which raises the power of x instead of lowering it.

6.12 Integrating Using Linear Partial Fractions

The integral $\displaystyle\int \frac{1}{x^2 - 1}\, dx$ is neither a u-substitution (the numerator is not a multiple of $2x$) nor an arctangent (the sign is wrong). But $x^2 - 1 = (x - 1)(x + 1)$ factors into *distinct linear factors*, and the

fraction can be split into two simpler fractions, each of which integrates to a logarithm:

$$\frac{1}{x^2-1} = \frac{1/2}{x-1} - \frac{1/2}{x+1}.$$

This splitting is called **partial fraction decomposition**. On the AP Calculus BC exam it is required only when the denominator factors into **nonrepeating linear factors**.

Linear partial fractions

Let $\dfrac{P(x)}{Q(x)}$ be a rational function with $\deg P < \deg Q$, where $Q(x) = (x-a_1)(x-a_2)\cdots(x-a_n)$ has distinct real roots. Then there are constants $A_1, \dots, A_n$ with

$$\frac{P(x)}{(x-a_1)(x-a_2)\cdots(x-a_n)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_n}{x-a_n},$$

so that $\displaystyle\int \frac{P(x)}{Q(x)}\,dx = \sum_{k=1}^{n} A_k \ln|x-a_k| + C$. **Finding the constants.** Multiply both sides by $Q(x)$ to clear denominators, then substitute each root $x = a_k$: every term but one vanishes, leaving A_k directly. (Equivalently, *cover up* the factor $(x-a_k)$ on the left and evaluate the rest at $x = a_k$.)

If $\deg P \geq \deg Q$, first use polynomial long division to write $\dfrac{P}{Q} = \text{polynomial} + \dfrac{R}{Q}$ with $\deg R < \deg Q$, then decompose $\dfrac{R}{Q}$.

VN **GIẢI THÍCH TIẾNG VIỆT**

Tách phân thức (partial fractions): khi mẫu số phân tích được thành tích các nhân tử bậc nhất *phân biệt* $(x-a_1)(x-a_2)\cdots$, ta viết phân thức thành tổng $\frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots$, rồi mỗi số hạng cho một $\ln|x-a_k|$. Cách tìm hằng số nhanh nhất: nhân hai vế với mẫu chung rồi **thay lần lượt các nghiệm** $x = a_k$ (mỗi lần chỉ còn lại một hằng số) — còn gọi là "che nhân tử" (cover-up). Nhớ hai điều: (1) nếu bậc tử $\geq$ bậc mẫu thì phải **chia đa thức trước**; (2) đề BC chỉ yêu cầu mẫu có nhân tử bậc nhất không lặp — các mẫu như x^2+1 (vô nghiệm) dẫn tới arctan chứ không tách được.

Ví dụ mẫu · Two linear factors

Evaluate $\displaystyle\int \frac{5x-4}{(x-2)(x+1)}\,dx$.

Set up: $\dfrac{5x-4}{(x-2)(x+1)} = \dfrac{A}{x-2} + \dfrac{B}{x+1}$. Multiply through by $(x-2)(x+1)$:

$$5x-4 = A(x+1) + B(x-2).$$

Substitute the roots. $x=2$: $6 = 3A \Rightarrow A = 2$. $\quad x=-1$: $-9 = -3B \Rightarrow B = 3$.

Check (compare coefficients): $2(x+1) + 3(x-2) = 5x-4$. ✓

Integrate:

$$\int \left(\frac{2}{x-2} + \frac{3}{x+1}\right) dx = 2\ln|x-2| + 3\ln|x+1| + C.$$

Ví dụ mẫu · Three linear factors

Evaluate $\displaystyle\int \frac{4x^2-3x-4}{x^3+x^2-2x}\,dx$.

Factor the denominator: $x^3+x^2-2x = x(x^2+x-2) = x(x-1)(x+2)$ — three distinct linear factors. Write

$$\frac{4x^2-3x-4}{x(x-1)(x+2)} = \frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+2} \implies 4x^2-3x-4 = A(x-1)(x+2)+Bx(x+2)+Cx(x-1).$$

$x=0$: $-4 = A(-1)(2) \Rightarrow A = 2$. $x=1$: $4-3-4 = -3 = B(1)(3) \Rightarrow B = -1$. $x=-2$: $16+6-4 = 18 = C(-2)(-3) \Rightarrow C = 3$.

$$\int \frac{4x^2-3x-4}{x^3+x^2-2x}\,dx = 2\ln|x| - \ln|x-1| + 3\ln|x+2| + C.$$

Ví dụ mẫu · Divide first

Evaluate $\displaystyle\int \frac{x^3+2x}{x^2-x-2}\,dx$.

The numerator has degree $3 \geq 2$, so divide first. Since $(x+1)(x^2-x-2) = x^3-3x-2$, the remainder is $(x^3+2x)-(x^3-3x-2) = 5x+2$:

$$\frac{x^3+2x}{x^2-x-2} = x+1+\frac{5x+2}{(x-2)(x+1)}.$$

Decompose the remainder: $5x+2 = A(x+1)+B(x-2)$. $x=2$: $12 = 3A \Rightarrow A = 4$. $x=-1$: $-3 = -3B \Rightarrow B = 1$.

$$\int \frac{x^3+2x}{x^2-x-2}\,dx = \int\left(x+1+\frac{4}{x-2}+\frac{1}{x+1}\right)dx = \frac{x^2}{2}+x+4\ln|x-2|+\ln|x+1|+C.$$

Ví dụ mẫu · A definite integral and the log rules

Evaluate $\displaystyle\int_2^3 \frac{1}{x^2-1}\,dx$ exactly.

$\dfrac{1}{(x-1)(x+1)} = \dfrac{A}{x-1}+\dfrac{B}{x+1}$ with $1 = A(x+1)+B(x-1)$: $x=1$ gives $A = \frac{1}{2}$; $x=-1$ gives $B = -\frac{1}{2}$. Therefore

$$\int_2^3 \frac{dx}{x^2-1} = \frac{1}{2}\Big[\ln|x-1|-\ln|x+1|\Big]_2^3 = \frac{1}{2}\Big[\ln\Big|\frac{x-1}{x+1}\Big|\Big]_2^3 = \frac{1}{2}\left(\ln\frac{2}{4}-\ln\frac{1}{3}\right) = \frac{1}{2}\ln\frac{3}{2}.$$

Combining logarithms with $\ln a - \ln b = \ln\frac{a}{b}$ *before* substituting keeps the arithmetic short. ($\frac{1}{2}\ln\frac{3}{2} \approx 0.203$.)

Integrand	Denominator	Technique and result
$\dfrac{x}{x^2-4}$	numerator is $\frac{1}{2}(x^2-4)'$	u-sub: $\frac{1}{2}\ln\lvert x^2-4\rvert + C$
$\dfrac{1}{x^2-4}$	$(x-2)(x+2)$, distinct linear	partial fractions: $\frac{1}{4}\ln\left\lvert\dfrac{x-2}{x+2}\right\rvert + C$
$\dfrac{1}{x^2+4}$	irreducible (no real roots)	arctangent: $\frac{1}{2}\arctan\dfrac{x}{2} + C$
$\dfrac{x}{x^2+4}$	irreducible, numerator matches	u-sub: $\frac{1}{2}\ln(x^2+4) + C$

Four integrands that look alike but call for three different techniques. Always check for a u-substitution before reaching for partial fractions.

(!) Lỗi thường gặp: Partial fractions apply only to *proper* rational functions. Applying the decomposition to $\dfrac{x^3+2x}{x^2-x-2}$ without dividing first produces constants that cannot satisfy the identity, because no combination $\frac{A}{x-2}+\frac{B}{x+1}$ can ever grow like x. A second common slip is integrating $\dfrac{A}{x-a}$ as $A\ln(x-a)$ without the absolute value, which loses the part of the domain where $x<a$.

6.13 Evaluating Improper Integrals

The definite integral $\int_a^b f(x)\,dx$ was defined for a *bounded* function on a *finite* interval. An **improper integral** relaxes one of those two requirements: either an endpoint is infinite, or the integrand becomes unbounded (has a vertical asymptote) at some point of $[a,b]$. In both cases the integral is defined as a **limit** of ordinary definite integrals.

6.13.1 Infinite limits of integration

Improper integrals with infinite limits

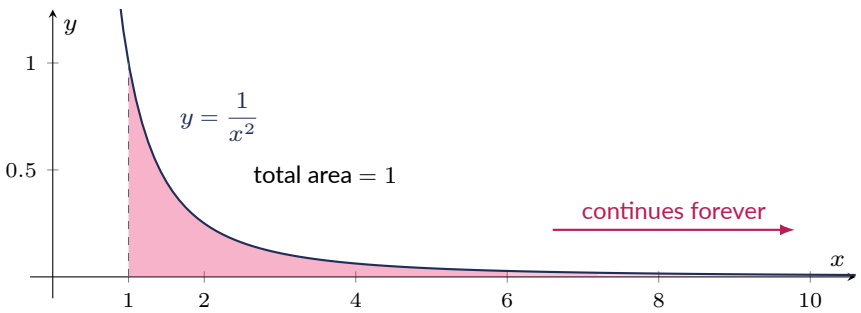
If f is continuous on the indicated interval,

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx, \qquad \int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$

If the limit exists as a finite number, the improper integral **converges** to that number; otherwise it **diverges**. For a doubly infinite integral, split at any convenient point c:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx,$$

which converges only if *both* pieces converge.



The region under $y=1/x^2$ for $x\ge 1$ is unbounded in extent, yet its area is finite: $\int_1^b x^{-2}\,dx = 1-\frac{1}{b}\to 1$ as $b\to\infty$.

Ví dụ mẫu · Convergent versus divergent

Determine whether each integral converges or diverges; evaluate it if it converges.

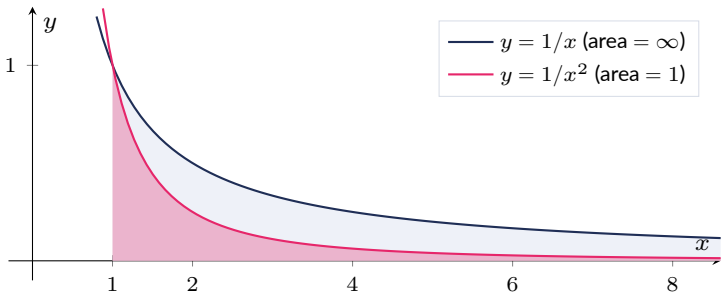
(a) $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ **(b)** $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ **(c)** $\displaystyle\int_0^\infty e^{-2x}\,dx$

(a) $\displaystyle\int_1^\infty \frac{dx}{x^2} = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty}\left(1-\frac{1}{b}\right) = 1$. **Converges** to 1.

(b) $\displaystyle\int_1^\infty \frac{dx}{x} = \lim_{b\to\infty}\left[\ln x\right]_1^b = \lim_{b\to\infty}\ln b = \infty$. **Diverges.**

(c) $\displaystyle\int_0^\infty e^{-2x}\,dx = \lim_{b\to\infty}\left[-\tfrac{1}{2}e^{-2x}\right]_0^b = \lim_{b\to\infty}\left(\tfrac{1}{2}-\tfrac{1}{2}e^{-2b}\right) = \tfrac{1}{2}$. **Converges** to $\frac{1}{2}$.

Parts (a) and (b) show how delicate the question is: $\frac{1}{x}$ and $\frac{1}{x^2}$ both approach 0, but $\frac{1}{x}$ does not approach 0 fast enough for the area to be finite.



On $[1,\infty)$, the extra navy sliver between $1/x$ and $1/x^2$ already has infinite area. The rate at which $f(x)\to 0$, not merely the fact that it does, decides convergence.

6.13.2 The p-integrals

Examples (a) and (b) are the two most important members of a family that serves as a benchmark for every other improper integral.

p-integrals (with p a constant):

$$\int_1^\infty \frac{dx}{x^p} \text{ converges } \iff p>1 \quad \left(\text{value } \frac{1}{p-1}\right);$$

$$\int_0^1 \frac{dx}{x^p} \text{ converges } \iff p<1 \quad \left(\text{value } \frac{1}{1-p}\right).$$

At $p=1$ *both* diverge. Derivation for $p\neq 1$: $\displaystyle\int_1^b x^{-p}\,dx = \frac{b^{1-p}-1}{1-p}$, and $b^{1-p}\to 0$ as $b\to\infty$ exactly when $1-p<0$.

VN **GIẢI THÍCH TIẾNG VIỆT**

Tích phân suy rộng được định nghĩa bằng **giới hạn**: thay cận vô hạn (hoặc điểm mà hàm không bị chặn) bằng một biến b, tính tích phân xác định thông thường, rồi cho b tiến tới giới hạn. Nếu giới hạn là một số hữu hạn thì tích phân **hội tụ**, ngược lại thì **phân kỳ**. Mốc so sánh quan trọng nhất là **tích phân** p: $\int_1^\infty \frac{dx}{x^p}$ hội tụ khi và chỉ khi $p>1$; còn $\int_0^1 \frac{dx}{x^p}$ hội tụ khi và chỉ khi $p<1$. Với $p=1$ cả hai đều phân kỳ. Trên đề AP, phải viết rõ ký hiệu giới hạn $\lim_{b\to\infty}$ — viết thẳng "∞" vào nguyên hàm sẽ bị trừ điểm.

Ví dụ mẫu · Integration by parts inside a limit

Evaluate $\displaystyle\int_0^\infty xe^{-x}\,dx$.

First find an antiderivative by parts: $u = x$, $dv = e^{-x}\,dx \Rightarrow du = dx$, $v = -e^{-x}$:

$$\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C = -(x+1)e^{-x} + C.$$

Then

$$\int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty}\Big[-(x+1)e^{-x}\Big]_0^b = \lim_{b\to\infty}\left(1 - \frac{b+1}{e^b}\right).$$

The fraction $\dfrac{b+1}{e^b}$ has the indeterminate form $\dfrac{\infty}{\infty}$; by L'Hôpital's Rule, $\displaystyle\lim_{b\to\infty}\frac{b+1}{e^b} = \lim_{b\to\infty}\frac{1}{e^b} = 0$.

The integral **converges to** 1.

Ví dụ mẫu · A doubly infinite integral

Evaluate $\displaystyle\int_{-\infty}^\infty \frac{dx}{1+x^2}$.

Split at 0 and treat each half separately:

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty}\Big[\arctan x\Big]_0^b = \lim_{b\to\infty}\arctan b = \frac{\pi}{2},$$

$$\int_{-\infty}^0 \frac{dx}{1+x^2} = \lim_{a\to-\infty}(0 - \arctan a) = \frac{\pi}{2}.$$

Both halves converge, so the integral converges to $\dfrac{\pi}{2} + \dfrac{\pi}{2} = \pi$.

Ví dụ mẫu · Partial fractions on an infinite interval

Evaluate $\displaystyle\int_2^\infty \frac{dx}{x^2-1}$.

From the previous section, an antiderivative is $\dfrac{1}{2}\ln\left|\dfrac{x-1}{x+1}\right|$. Since $\dfrac{b-1}{b+1} \to 1$ and $\ln 1 = 0$,

$$\int_2^\infty \frac{dx}{x^2-1} = \lim_{b\to\infty}\frac{1}{2}\left(\ln\frac{b-1}{b+1} - \ln\frac{1}{3}\right) = \frac{1}{2}(0 - \ln\tfrac{1}{3}) = \frac{1}{2}\ln 3 \approx 0.549.$$

Notice that *separately* $\int_2^\infty \frac{dx}{x-1}$ and $\int_2^\infty \frac{dx}{x+1}$ both diverge; it is essential to combine the logarithms before taking the limit.

6.13.3 Integrands that are unbounded

If f has a vertical asymptote at an endpoint or inside the interval, the Fundamental Theorem cannot be applied across that point directly. Again we approach the trouble spot with a limit.

Improper integrals with unbounded integrands

If f is continuous on $[a,b)$ and unbounded near b: $\displaystyle\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx.$

If f is continuous on $(a,b]$ and unbounded near a: $\displaystyle\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx.$

If f is unbounded near an *interior* point c of (a,b), split there,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

and the integral converges only when both pieces converge.

Ví dụ mẫu · Asymptote at an endpoint

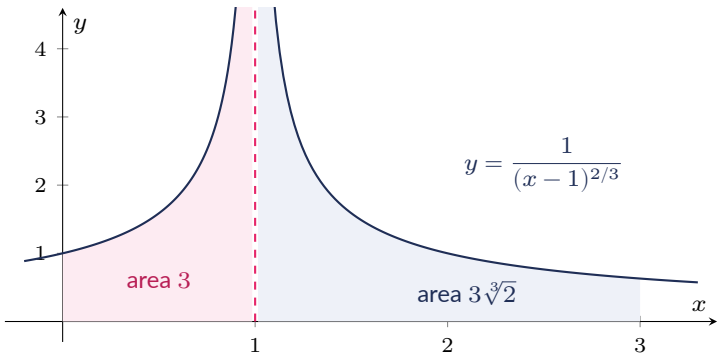
Evaluate **(a)** $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$ and **(b)** $\displaystyle\int_0^1 \ln x\,dx$.

(a) $\dfrac{1}{\sqrt{x}} \to \infty$ as $x \to 0^+$. $\displaystyle\int_0^1 x^{-1/2}\,dx = \lim_{t\to 0^+}\Big[2\sqrt{x}\Big]_t^1 = \lim_{t\to 0^+}(2-2\sqrt{t}) = 2.$ Converges (a p-integral on $[0,1]$ with $p=\frac{1}{2}<1$).

(b) $\ln x \to -\infty$ as $x\to 0^+$. Using $\int \ln x\,dx = x\ln x - x$:

$$\int_0^1 \ln x\,dx = \lim_{t\to 0^+}\Big[x\ln x - x\Big]_t^1 = (0-1) - \lim_{t\to 0^+}(t\ln t - t).$$

Write $t\ln t = \dfrac{\ln t}{1/t}$ (form $\frac{-\infty}{\infty}$); L'Hôpital gives $\displaystyle\lim_{t\to 0^+}\frac{1/t}{-1/t^2} = \lim_{t\to 0^+}(-t) = 0$. So the integral **converges to** -1 (negative, since $\ln x<0$ on $(0,1)$).



An interior asymptote at $x=1$: the integral over $[0,3]$ must be split there. Each piece converges, so $\int_0^3 (x-1)^{-2/3}\,dx = 3+3\sqrt[3]{2} \approx 6.780$.

Ví dụ mẫu · Asymptote inside the interval

Evaluate **(a)** $\displaystyle\int_0^3 \frac{dx}{(x-1)^{2/3}}$ and **(b)** $\displaystyle\int_{-1}^1 \frac{dx}{x^2}$.

(a) The integrand is unbounded at $x = 1 \in (0, 3)$. An antiderivative is $3(x-1)^{1/3}$.

$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{t\to 1^-} \Big[3(x-1)^{1/3}\Big]_0^t = 0 - 3(-1) = 3,$$

$$\int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{t\to 1^+} \Big[3(x-1)^{1/3}\Big]_t^3 = 3\sqrt[3]{2}.$$

Both converge, so the integral **converges to** $3 + 3\sqrt[3]{2} \approx 6.780$.

(b) The integrand is unbounded at $x = 0$. On the right half,

$$\int_0^1 \frac{dx}{x^2} = \lim_{t\to 0^+} \Big[-\frac{1}{x}\Big]_t^1 = \lim_{t\to 0^+} \left(-1 + \frac{1}{t}\right) = \infty,$$

so the whole integral **diverges** (one divergent piece is enough). Ignoring the asymptote and computing $\big[-\frac{1}{x}\big]_{-1}^{1} = -2$ gives a *negative* "area" for a *positive* integrand — a sure sign that the Fundamental Theorem was misapplied.

(!) Lỗi thường gặp: Before evaluating any definite integral, scan the interval for points where the integrand is undefined (zeros of a denominator, ln at 0, tan at $\frac{\pi}{2}$, ...). The Fundamental Theorem requires continuity on the *whole* closed interval; across a vertical asymptote it produces a number that is simply wrong, as in part (b) above.

6.13.4 Comparing improper integrals visually

Sometimes an antiderivative is unavailable, but convergence can still be decided by comparing graphs. If $0 \le f(x) \le g(x)$ for $x \ge a$, the region under f fits inside the region under g. Hence:

- if $\int_a^\infty g(x)\,dx$ *converges*, so does $\int_a^\infty f(x)\,dx$ (a smaller region inside a finite one);
- if $\int_a^\infty f(x)\,dx$ *diverges*, so does $\int_a^\infty g(x)\,dx$ (a larger region containing an infinite one).

Comparing with a p-integral is the usual move. This reasoning reappears in Unit 10 as the comparison test for series.

Ví dụ mẫu · Deciding by comparison

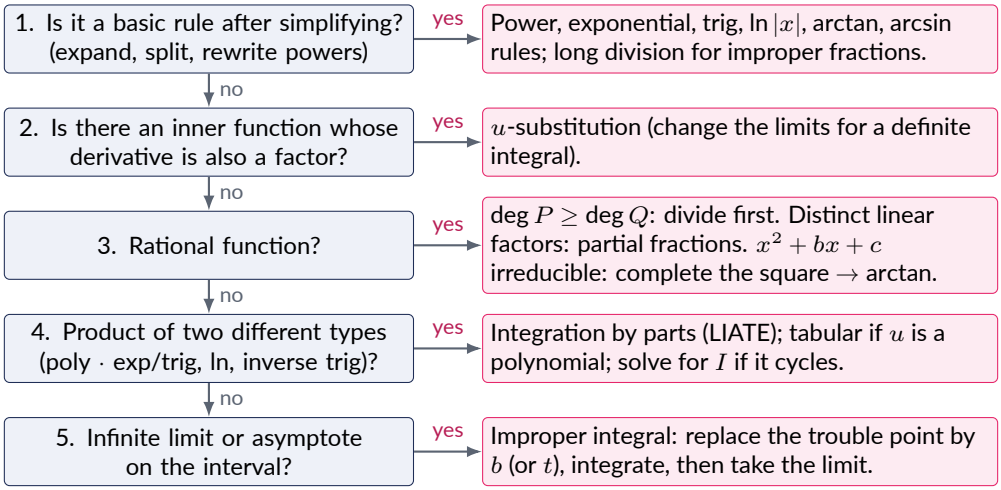
(a) For $x \ge 1$, $0 < \dfrac{1}{x^2+x} < \dfrac{1}{x^2}$. Since $\int_1^\infty \frac{dx}{x^2}$ converges, $\displaystyle\int_1^\infty \frac{dx}{x^2+x}$ converges. (Partial fractions confirm it: $\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$, and the value is $\displaystyle\lim_{b\to\infty}\Big[\ln\frac{x}{x+1}\Big]_1^b = 0 - \ln\frac{1}{2} = \ln 2$.)

(b) For $x \ge 1$, $e^{-x^2} \le e^{-x}$ (because $x^2 \ge x$). Since $\int_1^\infty e^{-x}\,dx = e^{-1}$ converges, $\displaystyle\int_1^\infty e^{-x^2}\,dx$ converges — even though e^{-x^2} has no elementary antiderivative.

(c) For $x \ge 2$, $\dfrac{1}{\sqrt{x}-1} > \dfrac{1}{\sqrt{x}} > 0$. Since $\int_2^\infty x^{-1/2}\,dx$ diverges ($p = \frac{1}{2} \le 1$), $\displaystyle\int_2^\infty \frac{dx}{\sqrt{x}-1}$ diverges.

6.14 Selecting Techniques for Antidifferentiation

On the exam, an integral does not come labeled with its technique. Topic 6.14 of the Course and Exam Description asks you to recognize the structure of an integrand and choose among the methods of this unit. A quick, systematic scan saves time and prevents the most expensive mistake: committing to a long method when a one-line one exists.



A decision sequence for antiderivatives. Step 5 is not an alternative to steps 1–4: an improper integral still needs an antiderivative from steps 1–4, followed by a limit.

VN GIẢI THÍCH TIẾNG VIỆT

Khi gặp một tích phân bất kỳ, hãy rà soát theo thứ tự: (1) rút gọn/khai triển xem có ra ngay quy tắc cơ bản không; (2) có "hàm trong" mà đạo hàm của nó cũng xuất hiện không → đổi biến u; (3) là phân thức hữu tỉ → chia đa thức nếu cần, tách phân thức nếu mẫu là tích nhân tử bậc nhất phân biệt, hoặc **bình phương đủ** (completing the square) để ra arctan nếu mẫu bậc hai vô nghiệm; (4) là tích của hai loại hàm khác nhau → từng phần; (5) nếu cận vô hạn hoặc hàm không bị chặn trên đoạn → tích phân suy rộng (vẫn phải tìm nguyên hàm bằng các bước trên rồi lấy giới hạn). Thói quen tốt: luôn thử đổi biến u trước khi dùng các kỹ thuật dài hơn.

6.14.1 Completing the square

A quadratic denominator with no real roots cannot be split into linear factors. Completing the square turns it into the form $(x-h)^2+a^2$, which matches $\displaystyle\int \frac{du}{u^2+a^2} = \frac{1}{a}\arctan\frac{u}{a}+C$.

Ví dụ mẫu · Completing the square

Evaluate $\displaystyle\int \frac{dx}{x^2+2x+5}$.

The discriminant $2^2-4(5)=-16<0$, so the denominator does not factor over the reals. Complete the square: $x^2+2x+5=(x+1)^2+4$. With $u=x+1$, $du=dx$:

$$\int \frac{dx}{(x+1)^2+2^2} = \frac{1}{2}\arctan\frac{x+1}{2}+C.$$

6.14.2 Mixed examples

Ví dụ mẫu · Choose the technique

Evaluate each integral, stating the technique.

(a) $\displaystyle\int \frac{2x+3}{x^2+1}\,dx$ **(b)** $\displaystyle\int \frac{x}{x^2-5x+6}\,dx$ **(c)** $\displaystyle\int x^2 \ln x\,dx$ **(d)** $\displaystyle\int e^{\sqrt{x}}\,dx$

(a) Split, then two basic rules. $\displaystyle\int \frac{2x}{x^2+1}\,dx + \int \frac{3}{x^2+1}\,dx = \ln(x^2+1) + 3\arctan x + C$. The first piece is a u-substitution with $u = x^2+1$; the second is an arctangent. Partial fractions do *not* apply (x^2+1 is irreducible).

(b) Partial fractions. $x^2-5x+6 = (x-2)(x-3)$ and the numerator is not a multiple of $2x-5$. From $x = A(x-3)+B(x-2)$: $x=2$ gives $A=-2$; $x=3$ gives $B=3$:

$$\int \frac{x\,dx}{(x-2)(x-3)} = -2\ln|x-2| + 3\ln|x-3| + C.$$

(c) Parts with $u = \ln x$ (L before A), $dv = x^2\,dx$, $du = \frac{dx}{x}$, $v = \frac{x^3}{3}$:

$$\int x^2 \ln x\,dx = \frac{x^3}{3}\ln x - \int \frac{x^2}{3}\,dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C.$$

(d) Substitution, then parts. Let $w = \sqrt{x}$, so $x = w^2$ and $dx = 2w\,dw$: $\displaystyle\int e^{\sqrt{x}}\,dx = \int 2we^w\,dw$. By parts ($u = 2w$, $dv = e^w\,dw$): $2we^w - 2e^w + C$. Returning to x:

$$\int e^{\sqrt{x}}\,dx = 2e^{\sqrt{x}}(\sqrt{x}-1) + C.$$

Ví dụ mẫu · Same look, different methods

Evaluate **(a)** $\displaystyle\int xe^{x^2}\,dx$ and **(b)** $\displaystyle\int xe^{2x}\,dx$ and **(c)** $\displaystyle\int x\sqrt{x+1}\,dx$.

(a) The factor x is half the derivative of the exponent x^2: u-substitution, $u = x^2$, gives $\frac{1}{2}e^{x^2} + C$. Integration by parts would go nowhere here.

(b) The exponent's derivative is the constant 2, not x, so no substitution absorbs the factor x: integration by parts gives $\frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + C$ (first example of Section 6.11).

(c) Parts would work, but the substitution $w = x+1$ (so $x = w-1$, $dx = dw$) is quicker:

$$\int (w-1)w^{1/2}\,dw = \int (w^{3/2} - w^{1/2})\,dw = \tfrac{2}{5}(x+1)^{5/2} - \tfrac{2}{3}(x+1)^{3/2} + C.$$

Integrand pattern → first technique to try

Pattern	Technique
$f(g(x))\,g'(x)$ (up to a constant)	u-substitution, $u = g(x)$
$\dfrac{P(x)}{Q(x)}$, $\deg P \geq \deg Q$	long division first
$\dfrac{P(x)}{(x-a)(x-b)}$, $\deg P < 2$, $a \neq b$	linear partial fractions
$\dfrac{1}{x^2+bx+c}$ with $b^2 - 4c < 0$	complete the square, then arctan
$x^n e^{kx}$, $x^n \sin kx$, $x^n \cos kx$	parts (tabular), $u = x^n$
$x^n \ln x$, $\ln x$, $\arctan x$, $\arcsin x$	parts, $u =$ the log / inverse trig factor
$e^{ax} \sin bx$, $e^{ax} \cos bx$	parts twice, solve for the integral
limit $\pm\infty$, or integrand unbounded on $[a, b]$	improper integral: limit of a definite integral

6.15 BC Practice: Integration Techniques

Bài tập luyện tập

$$\int xe^{-x}\,dx =$$

(A) $-xe^{-x} + e^{-x} + C$

(B) $-xe^{-x} - e^{-x} + C$

(C) $\frac{1}{2}x^2e^{-x} + C$

(D) $xe^{-x} - e^{-x} + C$

Lời giải chi tiết

$u = x$, $dv = e^{-x}\,dx \Rightarrow du = dx$, $v = -e^{-x}$. Then $\int xe^{-x}\,dx = -xe^{-x} - \int(-e^{-x})\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C$. Check: $\frac{d}{dx}\left[-xe^{-x} - e^{-x}\right] = -e^{-x} + xe^{-x} + e^{-x} = xe^{-x}$. ✓ **(B)**. Choice (A) comes from writing $\int e^{-x}\,dx = e^{-x}$; choice (D) from taking $v = e^{-x}$ and also $\int e^{-x}\,dx = e^{-x}$.

Bài tập luyện tập

Evaluate $\displaystyle\int x^2 \ln x\,dx$ and $\displaystyle\int_1^e x^2 \ln x\,dx$.

Lời giải chi tiết

$u = \ln x$, $dv = x^2\,dx$, $du = \frac{1}{x}\,dx$, $v = \frac{x^3}{3}$: $\displaystyle\int x^2 \ln x\,dx = \frac{x^3}{3}\ln x - \int \frac{x^2}{3}\,dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C.$

Definite: $\left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right]_1^e = \left(\frac{e^3}{3} - \frac{e^3}{9}\right) - \left(0 - \frac{1}{9}\right) = \frac{2e^3}{9} + \frac{1}{9} = \dfrac{2e^3+1}{9} \approx 4.574.$

Bài tập luyện tập

$$\int_0^1 xe^{2x}\,dx =$$

(A) $\dfrac{e^2}{2}$

(B) $\dfrac{e^2-1}{4}$

(C) $\dfrac{e^2+1}{4}$

(D) $\dfrac{e^2+1}{2}$

Lời giải chi tiết

An antiderivative is $\frac{x}{2}e^{2x}-\frac{1}{4}e^{2x}$ (worked example in Section 6.11). Evaluate: $\left(\frac{e^2}{2}-\frac{e^2}{4}\right)-\left(0-\frac{1}{4}\right)=\frac{e^2}{4}+\frac{1}{4}=\dfrac{e^2+1}{4}$. **(C)**. Choice (B) forgets that the lower limit contributes $-(-\frac{1}{4})=+\frac{1}{4}$.

Bài tập luyện tập

$$\int x\sec^2 x\,dx=$$

(A) $x\tan x+\ln|\cos x|+C$

(B) $x\tan x-\ln|\cos x|+C$

(C) $\frac{1}{2}x^2\tan x+C$

(D) $x\tan x-\sec^2 x+C$

Lời giải chi tiết

$u=x$, $dv=\sec^2 x\,dx\Rightarrow du=dx$, $v=\tan x$. Then $\int x\sec^2 x\,dx=x\tan x-\int\tan x\,dx$. Since $\int\tan x\,dx=-\ln|\cos x|+C$, the result is $x\tan x+\ln|\cos x|+C$. **(A)**. Choice (B) uses the wrong sign for $\int\tan x\,dx$.

Bài tập luyện tập

Evaluate $\displaystyle\int e^{2x}\sin x\,dx$.

Lời giải chi tiết

Let $I=\int e^{2x}\sin x\,dx$. Keep u trigonometric both times.
First: $u=\sin x$, $dv=e^{2x}\,dx$, $du=\cos x\,dx$, $v=\frac{1}{2}e^{2x}$: $I=\frac{1}{2}e^{2x}\sin x-\frac{1}{2}\int e^{2x}\cos x\,dx$.
Second: $u=\cos x$, $dv=e^{2x}\,dx$, $du=-\sin x\,dx$, $v=\frac{1}{2}e^{2x}$: $\int e^{2x}\cos x\,dx=\frac{1}{2}e^{2x}\cos x+\frac{1}{2}I$.
Substitute: $I=\frac{1}{2}e^{2x}\sin x-\frac{1}{4}e^{2x}\cos x-\frac{1}{4}I$, so $\frac{5}{4}I=\frac{1}{4}e^{2x}(2\sin x-\cos x)$ and

$$\int e^{2x}\sin x\,dx=\frac{e^{2x}(2\sin x-\cos x)}{5}+C.$$

Check: derivative $=\frac{1}{5}\left[2e^{2x}(2\sin x-\cos x)+e^{2x}(2\cos x+\sin x)\right]=\frac{1}{5}e^{2x}(5\sin x)=e^{2x}\sin x$. ✓

Bài tập luyện tập

Use the tabular method to evaluate $\displaystyle\int x^2\cos(3x)\,dx$.

Lời giải chi tiết

D-column: $x^2,\ 2x,\ 2,\ 0$. I-column: $\cos 3x,\ \frac{1}{3}\sin 3x,\ -\frac{1}{9}\cos 3x,\ -\frac{1}{27}\sin 3x$. Diagonals with signs $+, -, +$:

$$\int x^2 \cos 3x\,dx = x^2\cdot\tfrac{1}{3}\sin 3x - 2x\cdot\left(-\tfrac{1}{9}\cos 3x\right) + 2\cdot\left(-\tfrac{1}{27}\sin 3x\right) + C$$
$$= \frac{x^2}{3}\sin 3x + \frac{2x}{9}\cos 3x - \frac{2}{27}\sin 3x + C.$$

Bài tập luyện tập

$\int \arcsin x\,dx =$

(A) $x\arcsin x - \sqrt{1-x^2} + C$

(B) $\dfrac{1}{\sqrt{1-x^2}} + C$

(C) $x\arcsin x + \sqrt{1-x^2} + C$

(D) $\frac{1}{2}(\arcsin x)^2 + C$

Lời giải chi tiết

$u = \arcsin x$, $dv = dx$, $du = \dfrac{dx}{\sqrt{1-x^2}}$, $v = x$: $\int \arcsin x\,dx = x\arcsin x - \int \dfrac{x}{\sqrt{1-x^2}}\,dx$. With $w = 1-x^2$, $dw = -2x\,dx$: $\int \dfrac{x\,dx}{\sqrt{1-x^2}} = -\dfrac{1}{2}\int w^{-1/2}\,dw = -\sqrt{1-x^2} + C$. Hence $x\arcsin x + \sqrt{1-x^2} + C$. **(C)**. Choice (A) loses the sign from $dw = -2x\,dx$.

Bài tập luyện tập

The function g has continuous derivatives, with $g(0) = 2$, $g(3) = -1$, $g'(0) = 4$, $g'(3) = 5$, and $\int_0^3 g(x)\,dx = 6$.

(a) Find $\int_0^3 x\,g'(x)\,dx$. **(b)** Find $\int_0^3 x\,g''(x)\,dx$. **(c)** Find $\int_0^3 (x^2+1)\,g'(x)\,dx$ given also $\int_0^3 x\,g(x)\,dx = 2$.

Lời giải chi tiết

(a) $u = x$, $dv = g'(x)\,dx$, $v = g(x)$: $\Big[xg(x)\Big]_0^3 - \int_0^3 g(x)\,dx = (3(-1) - 0) - 6 = -9$.

(b) $u = x$, $dv = g''(x)\,dx$, $v = g'(x)$: $\Big[xg'(x)\Big]_0^3 - \int_0^3 g'(x)\,dx = (3\cdot 5 - 0) - (g(3) - g(0)) = 15 - (-1-2) = 18$.

(c) $u = x^2+1$, $dv = g'(x)\,dx$, $du = 2x\,dx$, $v = g(x)$:

$$\Big[(x^2+1)g(x)\Big]_0^3 - \int_0^3 2x\,g(x)\,dx = (10(-1) - 1(2)) - 2(2) = -12 - 4 = -16.$$

Bài tập luyện tập

$$\int \frac{dx}{x^2-3x+2} =$$

(A) $\ln|x^2-3x+2|+C$

(B) $\ln\left|\frac{x-1}{x-2}\right|+C$

(C) $\ln|(x-1)(x-2)|+C$

(D) $\ln\left|\frac{x-2}{x-1}\right|+C$

Lời giải chi tiết

$x^2-3x+2=(x-1)(x-2)$. From $1=A(x-2)+B(x-1)$: $x=1$ gives $A=-1$; $x=2$ gives $B=1$. So $\int\left(\frac{1}{x-2}-\frac{1}{x-1}\right)dx=\ln|x-2|-\ln|x-1|+C=\ln\left|\frac{x-2}{x-1}\right|+C$. **(D)**. Choice (A) treats the integrand as if the numerator were the derivative of the denominator; choice (B) has the signs of A and B reversed.

Bài tập luyện tập

Evaluate $\int \frac{3x+11}{x^2-x-6}\,dx$.

Lời giải chi tiết

$x^2-x-6=(x-3)(x+2)$, and $3x+11=A(x+2)+B(x-3)$. $x=3$: $20=5A\Rightarrow A=4$. $x=-2$: $5=-5B\Rightarrow B=-1$. Check: $4(x+2)-(x-3)=3x+11$. ✓

$$\int \frac{3x+11}{x^2-x-6}\,dx=4\ln|x-3|-\ln|x+2|+C.$$

Bài tập luyện tập

Evaluate $\int \frac{2x^2+x-4}{x^2-x}\,dx$.

Lời giải chi tiết

Degrees are equal, so divide: $2x^2+x-4=2(x^2-x)+(3x-4)$, giving $2+\frac{3x-4}{x(x-1)}$. Decompose: $3x-4=A(x-1)+Bx$. $x=0$: $-4=-A\Rightarrow A=4$. $x=1$: $-1=B$. So

$$\int\left(2+\frac{4}{x}-\frac{1}{x-1}\right)dx=2x+4\ln|x|-\ln|x-1|+C.$$

Bài tập luyện tập

$$\int_3^5 \frac{dx}{x^2-4} =$$

(A) $\frac{1}{4}\ln\frac{15}{7}$

(B) $\frac{1}{2}\ln\frac{15}{7}$

(C) $\frac{1}{4}\ln\frac{7}{15}$

(D) $\frac{1}{4}\ln\frac{3}{7}$

Lời giải chi tiết

$\frac{1}{(x-2)(x+2)} = \frac{1/4}{x-2} - \frac{1/4}{x+2}$ (from $1 = A(x+2) + B(x-2)$ at $x = \pm 2$). Then $\frac{1}{4}\left[\ln\left|\frac{x-2}{x+2}\right|\right]_3^5 = \frac{1}{4}\left(\ln\frac{3}{7} - \ln\frac{1}{5}\right) = \frac{1}{4}\ln\frac{15}{7} \approx 0.191$. **(A)**. Choice (D) is only the value at the upper limit.

Bài tập luyện tập

Which of the following improper integrals converge?

I. $\int_1^\infty \frac{dx}{x^{3/2}}$ II. $\int_0^1 \frac{dx}{x^{3/2}}$ III. $\int_0^1 \frac{dx}{\sqrt{x}}$

(A) I only
(B) II only
(C) I and III only
(D) I, II, and III

Lời giải chi tiết

I: $p = \frac{3}{2} > 1$ on $[1, \infty)$ — converges (to 2). II: on $[0, 1]$ convergence needs $p < 1$, but $p = \frac{3}{2}$ — diverges ($\lim_{t\to 0^+}\left[-2x^{-1/2}\right]_t^1 = \infty$). III: $p = \frac{1}{2} < 1$ on $[0, 1]$ — converges (to 2). **(C)**.

Bài tập luyện tập

Evaluate $\int_0^\infty xe^{-x^2}\,dx$, or show that it diverges.

Lời giải chi tiết

With $u = x^2$, $\int xe^{-x^2}\,dx = -\frac{1}{2}e^{-x^2} + C$. Then $\int_0^\infty xe^{-x^2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_0^b = \lim_{b\to\infty}\left(\frac{1}{2} - \frac{1}{2}e^{-b^2}\right) = \frac{1}{2}$. Converges to $\frac{1}{2}$.

Bài tập luyện tập

Evaluate $\int_1^\infty \frac{\ln x}{x^2}\,dx$, or show that it diverges.

Lời giải chi tiết

By parts: $u = \ln x$, $dv = x^{-2}\,dx$, $du = \frac{1}{x}\,dx$, $v = -\frac{1}{x}$: $\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} + \int \frac{dx}{x^2} = -\frac{\ln x}{x} - \frac{1}{x} + C$.

$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\left(-\frac{\ln b}{b} - \frac{1}{b}\right) - (0 - 1)$. By L'Hôpital, $\lim_{b\to\infty}\frac{\ln b}{b} = \lim_{b\to\infty}\frac{1/b}{1} = 0$. The integral converges to 1.

Bài tập luyện tập

$\int_0^2 \dfrac{dx}{(x-1)^2}$ is

(A) -2
(C) 2
(B) 0
(D) divergent

Lời giải chi tiết

The integrand is unbounded at $x = 1 \in (0, 2)$, so split there. $\int_1^2 \dfrac{dx}{(x-1)^2} = \lim_{t\to 1^+} \left[-\dfrac{1}{x-1}\right]_t^2 = \lim_{t\to 1^+} \left(-1 + \dfrac{1}{t-1}\right) = \infty$. One piece diverges, so the integral **diverges**. **(D)**. Choice (A) is the result of applying the Fundamental Theorem across the asymptote — impossible anyway, since the integrand is positive.

Bài tập luyện tập

Determine whether each integral converges; evaluate those that do.
(a) $\int_e^\infty \dfrac{dx}{x(\ln x)^2}$ **(b)** $\int_e^\infty \dfrac{dx}{x \ln x}$

Lời giải chi tiết

Let $u = \ln x$, $du = \frac{dx}{x}$; when $x = e$, $u = 1$, and $u \to \infty$ as $x \to \infty$.
(a) $\int_1^\infty \dfrac{du}{u^2} = \lim_{b\to\infty} \left(1 - \dfrac{1}{b}\right) = 1$. Converges to 1.
(b) $\int_1^\infty \dfrac{du}{u} = \lim_{b\to\infty} \ln b = \infty$. Diverges. (These are p-integrals in disguise, with $p = 2$ and $p = 1$.)

Bài tập luyện tập

$\int_1^\infty \dfrac{dx}{x(x+1)} =$

(A) 1
(C) $-\ln 2$
(B) $\ln 2$
(D) divergent

Lời giải chi tiết

$\dfrac{1}{x(x+1)} = \dfrac{1}{x} - \dfrac{1}{x+1}$. Then $\lim_{b\to\infty} \left[\ln \dfrac{x}{x+1}\right]_1^b = \lim_{b\to\infty} \ln \dfrac{b}{b+1} - \ln \dfrac{1}{2} = 0 + \ln 2 = \ln 2$. **(B).** Choice (D) results from integrating $\frac{1}{x}$ and $\frac{1}{x+1}$ separately to ∞ and concluding $\infty - \infty$ diverges.

Bài tập luyện tập

Let R be the unbounded region in the first quadrant under the graph of $y = xe^{-x/2}$, for $x \geq 0$.
(a) Find the area of R.
(b) Find the volume of the solid generated when R is revolved about the x-axis.

Lời giải chi tiết

(a) By parts ($u = x$, $dv = e^{-x/2}\,dx$, $v = -2e^{-x/2}$): $\int xe^{-x/2}\,dx = -2xe^{-x/2} + 2\int e^{-x/2}\,dx = -2xe^{-x/2} - 4e^{-x/2} + C$.

$$\text{Area} = \lim_{b\to\infty}\Big[-(2x+4)e^{-x/2}\Big]_0^b = \lim_{b\to\infty}\left(4 - \frac{2b+4}{e^{b/2}}\right) = 4,$$

since $\dfrac{2b+4}{e^{b/2}} \to 0$ (L'Hôpital: $\dfrac{2}{\frac{1}{2}e^{b/2}} \to 0$).

(b) Disks: $V = \pi\displaystyle\int_0^\infty \left(xe^{-x/2}\right)^2 dx = \pi\int_0^\infty x^2e^{-x}\,dx$. Tabular: D: $x^2, 2x, 2, 0$; I: $e^{-x}, -e^{-x}, e^{-x}, -e^{-x}$, so $\int x^2e^{-x}\,dx = -e^{-x}(x^2+2x+2) + C$.

$$V = \pi\lim_{b\to\infty}\Big[-e^{-x}(x^2+2x+2)\Big]_0^b = \pi(0+2) = 2\pi \approx 6.283.$$

Bài tập luyện tập

$\displaystyle\int_0^1 \arctan x\,dx =$

(A) $\dfrac{\pi}{4}$

(B) $\dfrac{\pi}{4} + \dfrac{1}{2}\ln 2$

(C) $\dfrac{\pi}{4} - \ln 2$

(D) $\dfrac{\pi}{4} - \dfrac{1}{2}\ln 2$

Lời giải chi tiết

Using $\int \arctan x\,dx = x\arctan x - \frac{1}{2}\ln(1+x^2)$: $\Big[x\arctan x - \frac{1}{2}\ln(1+x^2)\Big]_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln 2 \approx 0.439$. **(D)**.

Bài tập luyện tập

For each integral, name the most efficient first technique (do not evaluate):

(a) $\displaystyle\int \frac{x^2}{x^3+1}\,dx$ **(b)** $\displaystyle\int \frac{dx}{x^2+6x+13}$ **(c)** $\displaystyle\int \frac{dx}{x^2+6x+8}$ **(d)** $\displaystyle\int x^3e^x\,dx$ **(e)** $\displaystyle\int x^2e^{x^3}\,dx$

Lời giải chi tiết

(a) u-substitution, $u = x^3+1$ (numerator is $\frac{1}{3}u'$): result $\frac{1}{3}\ln|x^3+1| + C$.
(b) Discriminant $36 - 52 < 0$: complete the square, $(x+3)^2 + 4$, giving $\frac{1}{2}\arctan\frac{x+3}{2} + C$.
(c) $x^2+6x+8 = (x+2)(x+4)$: partial fractions, giving $\frac{1}{2}\ln\left|\frac{x+2}{x+4}\right| + C$.
(d) Integration by parts three times — tabular with $u = x^3$: $e^x(x^3 - 3x^2 + 6x - 6) + C$.
(e) u-substitution, $u = x^3$: $\frac{1}{3}e^{x^3} + C$.

Bài tập luyện tập

Evaluate $\displaystyle\int x\sqrt{x+1}\,dx$.

Lời giải chi tiết

Let $w = x + 1$, so $x = w - 1$, $dx = dw$: $\int (w-1)\sqrt{w}\,dw = \int (w^{3/2} - w^{1/2})dw = \frac{2}{5}w^{5/2} - \frac{2}{3}w^{3/2} + C = \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C$. (By parts with $u = x$, $dv = \sqrt{x+1}\,dx$ gives an equivalent answer.)

Bài tập luyện tập

The function h is defined by $h(x) = \displaystyle\int_1^x \frac{dt}{t(t+2)}$ for $x \geq 1$.

(a) Find $h(x)$ explicitly. **(b)** Find $\lim\limits_{x\to\infty} h(x)$, and interpret it as an improper integral.

Lời giải chi tiết

(a) $\frac{1}{t(t+2)} = \frac{1/2}{t} - \frac{1/2}{t+2}$ (from $1 = A(t+2) + Bt$ at $t = 0$ and $t = -2$). So $h(x) = \frac{1}{2}\left[\ln\frac{t}{t+2}\right]_1^x = \frac{1}{2}\ln\frac{x}{x+2} - \frac{1}{2}\ln\frac{1}{3} = \frac{1}{2}\ln\frac{3x}{x+2}$.

(b) $\frac{3x}{x+2} \to 3$, so $\lim\limits_{x\to\infty} h(x) = \frac{1}{2}\ln 3 \approx 0.549$. This limit is exactly the definition of $\displaystyle\int_1^\infty \frac{dt}{t(t+2)}$, which therefore converges to $\frac{1}{2}\ln 3$.

6.16 Problem Bank

Bài tập luyện tập

A particle moves along a line with velocity $v(t) = 6 - 2t$ (m/s) for $0 \leq t \leq 3$. Using geometry, the particle's displacement over $[0, 3]$ is:

(A) 9 **(C)** 6

(B) 18 **(D)** 12

Lời giải chi tiết

$v(t)$ is linear with $v(0) = 6$ and $v(3) = 0$, and $v(t) \geq 0$ throughout $[0, 3]$, so the region under the graph is a triangle with base 3 and height 6. Area $= \frac{1}{2}(3)(6) = 9$. **(A)**.

Bài tập luyện tập

On an interval where a function f is decreasing, the left Riemann sum L_n and right Riemann sum R_n satisfy:

(A) L_n underestimates, R_n overestimates **(C)** Both overestimate

(B) L_n overestimates, R_n underestimates **(D)** Both underestimate

Lời giải chi tiết

On a decreasing interval, the left endpoint of each subinterval gives the largest function value there (rectangle sits above the curve, overestimate), while the right endpoint gives the smallest value (rectangle sits below the curve, underestimate). **(B)**.

Bài tập luyện tập

The rate of change $h(x)$ of a quantity is recorded at $x = 0, 3, 6, 9, 12$:

x	0	3	6	9	12
$h(x)$	2	5	9	14	20

Use a left Riemann sum with $n = 4$ and a right Riemann sum with $n = 4$ to estimate $\int_0^{12} h(x)\,dx$, and state which is the over- and which is the underestimate.

Lời giải chi tiết

$\Delta x = 3$. Left sum: $L_4 = 3(2 + 5 + 9 + 14) = 3(30) = 90$. Right sum: $R_4 = 3(5 + 9 + 14 + 20) = 3(48) = 144$. Since h is increasing throughout, $L_4 = 90$ **underestimates** and $R_4 = 144$ **overestimates** $\int_0^{12} h(x)\,dx$.

Bài tập luyện tập

Values of $p(x)$ are given at $x = 1.5, 4.5, 7.5, 10.5$, the midpoints of four subintervals of width 3 covering $[0, 12]$:

x	1.5	4.5	7.5	10.5
$p(x)$	3	6	11	15

Estimate $\int_0^{12} p(x)\,dx$ using a midpoint Riemann sum.

Lời giải chi tiết

Each subinterval has width $\Delta x = 3$, and the given values are already the midpoint values, so $M_4 = 3(3 + 6 + 11 + 15) = 3(35) = 105$.

Bài tập luyện tập

Using the data $x = 0, 2, 4, 6$ with $f(x) = 3, 7, 12, 20$, the Trapezoidal Rule estimate for $\int_0^{6} f(x)\,dx$ (with $n = 3$, $\Delta x = 2$) is:

(A) 61
(C) 55
(B) 44
(D) 70

Lời giải chi tiết

$T_3 = \frac{\Delta x}{2}[f(0) + 2f(2) + 2f(4) + f(6)] = \frac{2}{2}[3 + 2(7) + 2(12) + 20] = 1[3 + 14 + 24 + 20] = 61$.
(A).

Bài tập luyện tập

A rainfall-rate sensor $r(t)$ (mm/hr) is recorded every 2 hours:

t (hr)	0	2	4	6	8
$r(t)$ (mm/hr)	10	8	5	3	1

Compute L_4, R_4, and T_4 for $\displaystyle\int_0^8 r(t)\,dt$, and verify that $T_4 = \tfrac{1}{2}(L_4 + R_4)$.

Lời giải chi tiết

$\Delta t = 2$. $L_4 = 2(10 + 8 + 5 + 3) = 2(26) = 52$. $R_4 = 2(8 + 5 + 3 + 1) = 2(17) = 34$.

$$T_4 = \frac{2}{2}\big[10 + 2(8) + 2(5) + 2(3) + 1\big] = 1\big[10 + 16 + 10 + 6 + 1\big] = 43.$$

Check: $\tfrac{1}{2}(L_4 + R_4) = \tfrac{1}{2}(52 + 34) = \tfrac{1}{2}(86) = 43$, which matches T_4 exactly.

Bài tập luyện tập

The expression $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left(2 + \frac{3i}{n}\right)\frac{3}{n}$ is equal to which definite integral?

(A) $\displaystyle\int_0^3 (2 + x)\,dx$

(B) $\displaystyle\int_2^3 x\,dx$

(C) $\displaystyle\int_0^2 (3 + x)\,dx$

(D) $\displaystyle\int_0^3 2x\,dx$

Lời giải chi tiết

Comparing to $\sum f(x_i)\Delta x$ with $\Delta x = \frac{3}{n}$ (so $b - a = 3$) and right endpoints $x_i = \frac{3i}{n}$ on $[0, 3]$, the summand $2 + \frac{3i}{n} = 2 + x_i = f(x_i)$ identifies $f(x) = 2 + x$. So the limit equals $\displaystyle\int_0^3 (2 + x)\,dx$. **(A).**

Bài tập luyện tập

Evaluate $\displaystyle\int_0^1 x^3\,dx$ using the limit definition of a right Riemann sum. (Use $\displaystyle\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$.)

Lời giải chi tiết

With $\Delta x = \frac{1}{n}$ and $x_i = \frac{i}{n}$, $f(x_i) = \dfrac{i^3}{n^3}$, so

$$R_n = \sum_{i=1}^{n}\frac{i^3}{n^3}\cdot\frac{1}{n} = \frac{1}{n^4}\sum_{i=1}^{n} i^3 = \frac{1}{n^4}\left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2(n+1)^2}{4n^4} = \frac{(n+1)^2}{4n^2}.$$

As $n \to \infty$, $\dfrac{(n+1)^2}{n^2} \to 1$, so $\displaystyle\int_0^1 x^3\,dx = \lim_{n\to\infty} R_n = \frac{1}{4}$. (Matches FTC2: $\left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4}$.)

Bài tập luyện tập

If $\displaystyle\int_2^6 f(x)\,dx = 10$ and $\displaystyle\int_2^6 g(x)\,dx = -3$, then $\displaystyle\int_2^6 [4f(x) + g(x)]\,dx =$

(A) 37
(C) 43
(B) 7
(D) 13

Lời giải chi tiết

By linearity: $4(10) + (-3) = 40 - 3 = 37$. **(A)**.

Bài tập luyện tập

Given $\displaystyle\int_1^4 f(x)\,dx = 9$ and $\displaystyle\int_1^7 f(x)\,dx = 20$, find $\displaystyle\int_4^7 f(x)\,dx$ and $\displaystyle\int_7^1 f(x)\,dx$.

Lời giải chi tiết

By additivity, $\displaystyle\int_4^7 f(x)\,dx = \int_1^7 f(x)\,dx - \int_1^4 f(x)\,dx = 20 - 9 = 11$. By reversing the limits, $\displaystyle\int_7^1 f(x)\,dx = -\int_1^7 f(x)\,dx = -20$.

Bài tập luyện tập

If $3 \le f(x) \le 7$ for all x in $[2, 9]$, which inequality must be true for $\displaystyle\int_2^9 f(x)\,dx$?

(A) $21 \le \displaystyle\int_2^9 f(x)\,dx \le 49$
(C) $14 \le \displaystyle\int_2^9 f(x)\,dx \le 42$
(B) $3 \le \displaystyle\int_2^9 f(x)\,dx \le 7$
(D) $\displaystyle\int_2^9 f(x)\,dx = 35$ exactly

Lời giải chi tiết

The comparison property gives $m(b-a) \le \int_a^b f \le M(b-a)$ with $m = 3$, $M = 7$, $b - a = 7$: $3(7) \le \int \le 7(7)$, i.e., $21 \le \int_2^9 f(x)\,dx \le 49$. **(A)**.

Bài tập luyện tập

Let $F(x) = \displaystyle\int_0^x (2t^2 - 7)\,dt$. Find $F'(x)$ and evaluate $F'(4)$.

Lời giải chi tiết

By FTC1, $F'(x) = 2x^2 - 7$. So $F'(4) = 2(16) - 7 = 32 - 7 = 25$.

Bài tập luyện tập

If $G(x) = \displaystyle\int_3^{x^3} \cos(t)\,dt$, then $G'(x) =$

(A) $3x^2\cos(x^3)$ **(C)** $\cos(3x^2)$

(B) $\cos(x^3)$ **(D)** $3x^2\cos(x)$

Lời giải chi tiết

By FTC1 with the chain rule, upper limit $g(x) = x^3$ with $g'(x) = 3x^2$: $G'(x) = \cos(x^3) \cdot 3x^2 = 3x^2\cos(x^3)$. **(A)**.

Bài tập luyện tập

Let $H(x) = \displaystyle\int_{2x}^{5} \sqrt{t^3+1}\,dt$. Find $H'(x)$.

Lời giải chi tiết

Reverse the limits first: $H(x) = -\displaystyle\int_5^{2x} \sqrt{t^3+1}\,dt$. By FTC1 with the chain rule (upper limit $2x$, derivative 2):

$$H'(x) = -\sqrt{(2x)^3+1} \cdot 2 = -2\sqrt{8x^3+1}.$$

Bài tập luyện tập

$\displaystyle\int_0^3 (4x-1)\,dx =$

(A) 15 **(C)** 9

(B) 12 **(D)** 18

Lời giải chi tiết

Antiderivative: $2x^2 - x$. Evaluate: $[2x^2 - x]_0^3 = (18-3) - (0) = 15$. **(A)**.

Bài tập luyện tập

Evaluate $\displaystyle\int_1^3 \left(2x^2 - \frac{1}{x^2}\right) dx$ and simplify completely.

Lời giải chi tiết

An antiderivative is $F(x) = \frac{2}{3}x^3 + \frac{1}{x}$ (since the antiderivative of $-x^{-2}$ is x^{-1}). Then

$$F(3) - F(1) = \left(\frac{2}{3}(27) + \frac{1}{3}\right) - \left(\frac{2}{3} + 1\right) = \left(18 + \frac{1}{3}\right) - \left(\frac{5}{3}\right) = \frac{55}{3} - \frac{5}{3} = \frac{50}{3}.$$

Bài tập luyện tập

$\int_0^{\pi} \sin x\, dx =$

(A) 2

(C) 1

(B) 0

(D) -2

Lời giải chi tiết

Antiderivative: $-\cos x$. Evaluate: $[-\cos x]_0^{\pi} = (-\cos\pi) - (-\cos 0) = (1) - (-1) = 2$. **(A).**

Bài tập luyện tập

Find $f(x)$ given $f''(x) = 6x - 4$, $f'(0) = 3$, and $f(0) = -2$.

Lời giải chi tiết

Integrate once: $f'(x) = 3x^2 - 4x + C_1$. Using $f'(0) = 3$: $C_1 = 3$, so $f'(x) = 3x^2 - 4x + 3$.
Integrate again: $f(x) = x^3 - 2x^2 + 3x + C_2$. Using $f(0) = -2$: $C_2 = -2$, so

$$f(x) = x^3 - 2x^2 + 3x - 2.$$

Bài tập luyện tập

$\int \left(\frac{4}{x} - 3\sin x\right) dx =$

(A) $4\ln|x| + 3\cos x + C$

(C) $-4\ln|x| + 3\cos x + C$

(B) $4\ln|x| - 3\cos x + C$

(D) $4\ln|x| + 3\sin x + C$

Lời giải chi tiết

$\int \frac{4}{x}\, dx = 4\ln|x|$; $\int (-3\sin x)\, dx = -3(-\cos x) = 3\cos x$. Sum: $4\ln|x| + 3\cos x + C$. **(A).**

Bài tập luyện tập

Evaluate $\int_2^5 \frac{3}{x}\, dx$, and explain why the reverse power rule cannot be applied directly to this integrand.

Lời giải chi tiết

An antiderivative of $\frac{3}{x}$ is $3\ln|x|$, so $\int_2^5 \frac{3}{x}\,dx = [3\ln|x|]_2^5 = 3\ln 5 - 3\ln 2 = 3\ln\left(\frac{5}{2}\right)$. The reverse power rule $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ requires dividing by $n+1$; here the integrand is x^{-1}, so $n = -1$ and $n+1 = 0$, making the formula undefined. The rule $\int \frac{1}{x}\,dx = \ln|x| + C$ must be used instead, since $\frac{d}{dx}\ln|x| = \frac{1}{x}$ for all $x \neq 0$.

Bài tập luyện tập

$\int 4x(x^2+3)^5\,dx =$

(A) $\frac{1}{3}(x^2+3)^6 + C$

(B) $\frac{1}{6}(x^2+3)^6 + C$

(C) $(x^2+3)^6 + C$

(D) $2(x^2+3)^6 + C$

Lời giải chi tiết

Let $u = x^2+3$, $du = 2x\,dx$, so $4x\,dx = 2\,du$. Then $\int u^5 \cdot 2\,du = 2\cdot\frac{u^6}{6} + C = \frac{1}{3}u^6 + C = \frac{1}{3}(x^2+3)^6 + C$. **(A)**.

Bài tập luyện tập

Evaluate $\int_0^2 x^2\sqrt{x^3+1}\,dx$ using u-substitution, converting the limits of integration.

Lời giải chi tiết

Let $u = x^3+1$, $du = 3x^2\,dx$, so $x^2\,dx = \frac{du}{3}$. Limits: $x=0 \Rightarrow u=1$; $x=2 \Rightarrow u=9$. Then

$$\int_0^2 x^2\sqrt{x^3+1}\,dx = \int_1^9 \sqrt{u}\cdot\frac{du}{3} = \frac{1}{3}\int_1^9 u^{1/2}\,du = \frac{1}{3}\left[\frac{2}{3}u^{3/2}\right]_1^9$$
$$= \frac{2}{9}(9^{3/2} - 1^{3/2}) = \frac{2}{9}(27-1) = \frac{2}{9}(26) = \frac{52}{9}.$$

Bài tập luyện tập

$\int_0^{\pi/4} \sec^2 x \tan x\,dx =$

(A) $\frac{1}{2}$

(B) 1

(C) $\frac{\pi}{4}$

(D) 0

Lời giải chi tiết

Let $u = \tan x$, $du = \sec^2 x\,dx$. Limits: $x = 0 \Rightarrow u = 0$; $x = \frac{\pi}{4} \Rightarrow u = 1$. Then $\displaystyle\int_0^1 u\,du = \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2}$. **(A)**.

Bài tập luyện tập

Evaluate $\displaystyle\int \frac{x+1}{\sqrt{x^2+2x+5}}\,dx$.

Lời giải chi tiết

Let $u = x^2 + 2x + 5$, so $du = (2x+2)\,dx = 2(x+1)\,dx$, giving $(x+1)\,dx = \frac{du}{2}$. Then

$$\int \frac{1}{\sqrt{u}} \cdot \frac{du}{2} = \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\cdot 2u^{1/2} + C = u^{1/2} + C = \sqrt{x^2+2x+5} + C.$$

Bài tập luyện tập

Rewriting $\frac{x^2-1}{x-3}$ using polynomial long division gives:

(A) $x + 3 + \frac{8}{x-3}$

(B) $x - 3 + \frac{8}{x-3}$

(C) $x + 3 - \frac{8}{x-3}$

(D) $x + 1 + \frac{2}{x-3}$

Lời giải chi tiết

Dividing: $(x-3)(x+3) = x^2 - 9$, and $(x^2-1) - (x^2-9) = 8$, so $x^2 - 1 = (x-3)(x+3) + 8$, giving $\frac{x^2-1}{x-3} = x + 3 + \frac{8}{x-3}$. **(A)**.

Bài tập luyện tập

Using the division from the previous problem, evaluate $\displaystyle\int_4^5 \frac{x^2-1}{x-3}\,dx$.

Lời giải chi tiết

From the previous problem, $\frac{x^2-1}{x-3} = x + 3 + \frac{8}{x-3}$, with antiderivative $F(x) = \frac{x^2}{2} + 3x + 8\ln|x-3|$. Then

$$\begin{aligned} F(5) - F(4) &= \left(\frac{25}{2} + 15 + 8\ln 2\right) - \left(\frac{16}{2} + 12 + 8\ln 1\right) \\ &= (27.5 + 8\ln 2) - (20 + 0) = 7.5 + 8\ln 2 = \frac{15}{2} + 8\ln 2. \end{aligned}$$

Bài tập luyện tập

$$\int \frac{x^3 - 2x^2 + 4}{x^3}\,dx =$$

(A) $x - 2\ln|x| - \dfrac{2}{x^2} + C$

(B) $x - 2\ln|x| + \dfrac{2}{x^2} + C$

(C) $x + 2\ln|x| - \dfrac{2}{x^2} + C$

(D) $1 - \dfrac{2}{x^2} - \dfrac{12}{x^4} + C$

Lời giải chi tiết

Split the fraction: $\dfrac{x^3 - 2x^2 + 4}{x^3} = 1 - \dfrac{2}{x} + 4x^{-3}$. Integrating: $x - 2\ln|x| + 4 \cdot \dfrac{x^{-2}}{-2} + C = x - 2\ln|x| - \dfrac{2}{x^2} + C$. **(A)**.

Bài tập luyện tập

Evaluate $\displaystyle\int_1^9 \left(5\sqrt{x} - \frac{4}{\sqrt{x}}\right) dx$.

Lời giải chi tiết

Rewrite as $5x^{1/2} - 4x^{-1/2}$, with antiderivative $F(x) = \dfrac{10}{3}x^{3/2} - 8x^{1/2}$. Using $9^{3/2} = 27$, $9^{1/2} = 3$, $1^{3/2} = 1^{1/2} = 1$:

$$F(9) - F(1) = \left(\frac{10}{3}(27) - 8(3)\right) - \left(\frac{10}{3}(1) - 8(1)\right) = (90 - 24) - \left(\frac{10}{3} - 8\right)$$
$$= 66 - \left(-\frac{14}{3}\right) = 66 + \frac{14}{3} = \frac{212}{3}.$$

Bài tập luyện tập

If $F(x) = \displaystyle\int_1^x \sqrt{t^2 + 1}\,dt$, then $F'(2) =$

(A) $\sqrt{5}$

(B) $\sqrt{3}$

(C) 5

(D) 2

Lời giải chi tiết

By FTC1, $F'(x) = \sqrt{x^2 + 1}$, so $F'(2) = \sqrt{4 + 1} = \sqrt{5}$. **(A)**.

Bài tập luyện tập

The rate of water flow into a tank, $R(t)$ (gal/min), is recorded every 4 minutes:

t (min)	0	4	8	12	16
$R(t)$ (gal/min)	20	26	31	35	38

(a) Use a right Riemann sum with 4 subintervals to approximate $\displaystyle\int_0^{16} R(t)\,dt$, and state whether

it over- or underestimates the total gallons that flowed in. (b) Use the Trapezoidal Rule with the same data to approximate $\int_0^{16} R(t)\,dt$.

Lời giải chi tiết

(a) $\Delta t = 4$. $R_4 = 4(26 + 31 + 35 + 38) = 4(130) = 520$ gallons. Since $R(t)$ is increasing throughout, the right endpoint of each subinterval gives the larger value, so R_4 **overestimates** the total volume.

(b)

$$T_4 = \frac{4}{2}[20 + 2(26) + 2(31) + 2(35) + 38] = 2[20 + 52 + 62 + 70 + 38] = 2(242) = 484 \text{ gallons.}$$

Bài tập luyện tập

Evaluate $\int_0^1 \frac{x}{x^2+4}\,dx$.

(A) $\frac{1}{2}\ln\left(\frac{5}{4}\right)$

(B) $\ln\left(\frac{5}{4}\right)$

(C) $\frac{1}{4}\ln 5$

(D) $\frac{1}{2}\ln 5$

Lời giải chi tiết

Let $u = x^2 + 4$, $du = 2x\,dx$, so $x\,dx = \frac{du}{2}$. Limits: $x = 0 \Rightarrow u = 4$; $x = 1 \Rightarrow u = 5$. Then

$$\int_4^5 \frac{1}{u} \cdot \frac{du}{2} = \frac{1}{2}\big[\ln|u|\big]_4^5 = \frac{1}{2}(\ln 5 - \ln 4) = \frac{1}{2}\ln\left(\frac{5}{4}\right).$$

(A).

Bài tập luyện tập

Given $\int_0^4 f(x)\,dx = 6$, find $\int_0^4 [5f(x) + 3]\,dx$.

Lời giải chi tiết

Split using linearity and the fact that $\int_0^4 3\,dx$ is the area of a rectangle of height 3 and width 4:

$$\int_0^4 [5f(x) + 3]\,dx = 5\int_0^4 f(x)\,dx + \int_0^4 3\,dx = 5(6) + 3(4) = 30 + 12 = 42.$$

Bài tập luyện tập

Evaluate $\int_0^2 |x-1|\,dx$ by splitting the integral at the point where the absolute value changes definition.

Lời giải chi tiết

For $x<1$, $|x-1|=1-x$; for $x\geq 1$, $|x-1|=x-1$. Split at $x=1$:

$$\int_0^2 |x-1|\,dx=\int_0^1 (1-x)\,dx+\int_1^2 (x-1)\,dx.$$

First piece: $[x-\frac{x^2}{2}]_0^1=(1-\frac{1}{2})-0=\frac{1}{2}$. Second piece: $[\frac{x^2}{2}-x]_1^2=(2-2)-(\frac{1}{2}-1)=0-(-\frac{1}{2})=\frac{1}{2}$. Total: $\frac{1}{2}+\frac{1}{2}=1$.

Bài tập luyện tập

Evaluate $\int_0^2 \sqrt{4-x^2}\,dx$ using a geometric interpretation rather than an antiderivative.

Lời giải chi tiết

The graph of $y=\sqrt{4-x^2}$ on $[0,2]$ is the upper-right quarter of the circle $x^2+y^2=4$ (radius 2, centered at the origin), since $x\geq 0$, $y\geq 0$ there. The area under this quarter-circle arc equals one quarter of the full circle's area: $\frac{1}{4}\pi(2)^2=\frac{1}{4}(4\pi)=\pi$.

Differential Equations

Mục tiêu Unit: Xây dựng phương trình vi phân từ phát biểu bằng lời; kiểm chứng một hàm số có phải là nghiệm của phương trình vi phân; vẽ và đọc trường độ dốc (slope field) để suy luận hình dạng nghiệm mà không cần giải phương trình; xấp xỉ nghiệm bằng phương pháp Euler; tìm nghiệm tổng quát bằng phương pháp tách biến (separation of variables), kể cả khi nghiệm phải để ở dạng ẩn (implicit); dùng điều kiện đầu để tìm nghiệm riêng; mô hình tăng trưởng – phân rã mũ $\frac{dy}{dt} = ky$, bán chu kỳ phóng xạ, và định luật làm nguội Newton; và giới thiệu định tính phương trình vi phân logistic cùng sức chứa môi trường (carrying capacity) và hình dạng đường cong chữ S.

7.1 Modeling Situations with Differential Equations

A **differential equation** is an equation that relates an unknown function to one or more of its derivatives. Instead of solving for a number, solving a differential equation means finding a *function* $y = g(x)$ whose derivative (or derivatives) make the equation true for every x in its domain. For example, $\frac{dy}{dx} = 2x$ is a differential equation whose solutions are all functions of the form $y = x^2 + C$ — an entire **family** of functions, one for each choice of the constant C. This family is called the **general solution**. If we additionally require the solution curve to pass through a specific point, such as $y(0) = 5$, exactly one member of the family satisfies that requirement; this single function is called a **particular solution**.

Differential equations are classified by their **order**: the order is the order of the highest derivative that appears. Every equation studied in this unit is **first order** — it involves $\frac{dy}{dx}$ (or $\frac{dy}{dt}$) but no higher derivative. An equation such as $\frac{d^2y}{dx^2} + 4y = 0$ would be second order and requires different techniques not covered here.

Vocabulary: *order* = highest derivative present; *general solution* = the whole family of functions satisfying the differential equation, written with an arbitrary constant C; *particular solution* = the one member of that family singled out by an *initial condition* $y(x_0) = y_0$.

This unit develops two complementary ways to work with a first-order differential equation once it is written down. The first is **graphical and numerical**: slope fields (Section 7.3) and Euler's Method (Section 7.5) work for virtually any first-order equation and never require solving anything symbolically — they trade an exact formula for a picture or a table of estimates. The second is **algebraic**: separation of variables (Sections 7.6–7.7) produces an exact formula for y, but only when the equation can be written in the special separable form $\frac{dy}{dx} = g(x)h(y)$. Recognizing which approach a given equation calls for is itself an important skill, since not every differential equation that arises from a modeling situation is separable.

VN **GIẢI THÍCH TIẾNG VIỆT**

Phương trình vi phân (differential equation) là một phương trình chứa đạo hàm của một hàm số chưa biết – thay vì tìm một con số, ta cần tìm cả một **hàm số** $y = g(x)$ sao cho đạo hàm của nó thoả mãn phương trình đã cho. **Cấp (order)** của phương trình vi phân là cấp cao nhất của

đạo hàm xuất hiện trong đó; chương này chỉ xét các phương trình **cấp một** (chỉ chứa $\frac{dy}{dx}$). Khi "dịch" một câu phát biểu bằng lời sang phương trình vi phân, cụm từ "tốc độ thay đổi của X tỉ lệ thuận với Y" luôn được viết thành $\frac{dX}{dt} = kY$, trong đó k là hằng số tỉ lệ (có thể dương hoặc âm tuỳ ngữ cảnh).

Many real situations are described first in words, and the key skill is translating a verbal statement about a *rate of change* into symbols. The phrase "the rate of change of X is proportional to Y" always becomes $\dfrac{dX}{dt} = kY$, where k is an unknown constant of proportionality (its sign and size are usually determined later from data). Watch for variations: "proportional to the square of," "proportional to the product of …and …," or "proportional to the difference between …and …" each change the right-hand side accordingly, but the left-hand side is always the derivative of the quantity that is changing.

Translating words into symbols

- "The rate of change of P is proportional to P itself" $\Rightarrow$ $\dfrac{dP}{dt} = kP$ (unrestricted growth/decay).
- "The rate of change of y is proportional to the product of y and $(N-y)$" $\Rightarrow$ $\dfrac{dy}{dt} = ky(N-y)$ (growth limited by a fixed total N).
- "The rate of change of T is proportional to the difference between T and a fixed value T_{amb}" $\Rightarrow$ $\dfrac{dT}{dt} = k(T - T_{amb})$ (approach to equilibrium).

Ví dụ mẫu · Translating verbal statements

Translate each statement into a differential equation.

(a) A population of fish P grows at a rate proportional to the current population.
(b) In a school of N students, a rumor spreads at a rate proportional to the product of the number of students who have heard it, y, and the number who have not, $(N - y)$.
(c) The temperature T of a metal rod changes at a rate proportional to the difference between its temperature and the temperature of the surrounding air, T_a.

Reasoning: in each case, identify the quantity that is changing (this becomes the derivative on the left) and identify exactly what that rate is said to be proportional to (this becomes the right-hand side, multiplied by an unknown constant k).

(a) $\dfrac{dP}{dt} = kP$ **(b)** $\dfrac{dy}{dt} = ky(N - y)$ **(c)** $\dfrac{dT}{dt} = k(T - T_a)$

These three equations reappear later in this unit: (a) leads to the exponential model, (b) is the *logistic* model, and (c) is Newton's Law of Cooling.

7.2 Verifying Solutions of Differential Equations

Given a candidate function $y(x)$, we can check whether it actually solves a given differential equation without solving anything: differentiate $y(x)$ to get $y'(x)$ (and $y''(x)$ if the equation is second order), substitute both y and its derivative(s) into the differential equation, and simplify. If the resulting statement is true for every x in the domain, $y(x)$ is a solution; if it is only true for isolated values of x, or never true, $y(x)$ is **not** a solution.

To verify $y(x)$ solves a differential equation: (1) compute $y'(x)$ (and higher derivatives if needed); (2) substitute y and y' into both sides of the equation; (3) simplify each side independently; (4) check that the two sides are *identical as functions of* x — not just equal at one point.

VN **GIẢI THÍCH TIẾNG VIỆT**

Để kiểm chứng một hàm số $y(x)$ có phải là nghiệm của một phương trình vi phân hay không, ta tính đạo hàm $y'(x)$ (và $y''(x)$ nếu cần), rồi thay cả y lẫn y' vào hai vế của phương trình. Nếu hai vế bằng nhau với **mọi** x trong miền xác định thì $y(x)$ là một nghiệm; nếu chỉ đúng tại một vài giá trị x riêng lẻ (hoặc không đúng chút nào) thì $y(x)$ **không** phải là nghiệm. Đây thuần tuý là một phép thay thế và rút gọn đại số, không cần giải phương trình.

Ví dụ mẫu · Verifying candidate solutions

Consider the differential equation $\dfrac{dy}{dx} = -2y + 6$.

(a) Verify that $y = Ce^{-2x} + 3$ is a solution for *every* constant C.
(b) Determine whether $y = e^{-2x} + 5$ is a solution.

(a) Differentiate: $\dfrac{dy}{dx} = -2Ce^{-2x}$ (the derivative of the constant 3 is 0). Substitute into the right-hand side:

$$-2y + 6 = -2(Ce^{-2x} + 3) + 6 = -2Ce^{-2x} - 6 + 6 = -2Ce^{-2x}.$$

This matches $\dfrac{dy}{dx} = -2Ce^{-2x}$ exactly, for *any* value of C — so $y = Ce^{-2x} + 3$ is a solution for every constant C; it is the general solution. (Verification is exactly the tool needed later in this unit whenever a separable equation produces an *implicit* solution that cannot be solved for y: instead of isolating y, we differentiate the implicit equation and confirm it reproduces the original differential equation, as in Section 7.7.)

(b) Differentiate: $\dfrac{dy}{dx} = -2e^{-2x}$. Substitute into the right-hand side:

$$-2y + 6 = -2(e^{-2x} + 5) + 6 = -2e^{-2x} - 10 + 6 = -2e^{-2x} - 4.$$

Comparing, $-2e^{-2x} \neq -2e^{-2x} - 4$ (the two sides differ by 4 for every x), so $y = e^{-2x} + 5$ is **not** a solution. The additive constant must be exactly 3 (the value that makes the right-hand side vanish, $-2(3) + 6 = 0$), not an arbitrary number — only the coefficient of e^{-2x} is free to vary.

7.3 Slope Fields

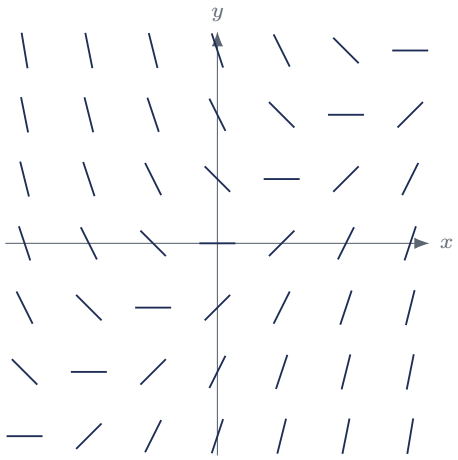
7.3.1 Sketching a Slope Field

A **slope field** (also called a *direction field*) is a picture of a differential equation $\dfrac{dy}{dx} = f(x, y)$ built entirely without solving it. At each point (x, y) on a grid, we evaluate $f(x, y)$ and draw a short line segment through that point with exactly that slope. Because $f(x, y)$ can be evaluated at any grid point using only arithmetic, a slope field can always be constructed even when the differential equation cannot be solved by any technique in this course.

Procedure: choose a grid of points (x, y); at each point, substitute the coordinates into the right-hand side $f(x, y)$ of $\dfrac{dy}{dx} = f(x, y)$ to get a numerical slope; draw a short segment of that slope centered at the point. Repeat for every grid point. The completed picture shows the direction every solution curve must travel as it passes through each point, without ever solving for y.

VN **GIẢI THÍCH TIẾNG VIỆT**

Trường độ dốc (slope field, còn gọi là direction field) là một tập hợp các đoạn thẳng ngắn vẽ tại nhiều điểm lưới (x, y), mỗi đoạn có độ dốc đúng bằng giá trị $\frac{dy}{dx}$ mà phương trình vi phân quy định tại điểm đó – hoàn toàn **không cần giải** phương trình. Để vẽ, ta chỉ cần thay toạ độ (x, y) của từng điểm lưới vào vế phải của phương trình để tính độ dốc, rồi vẽ một đoạn thẳng ngắn có độ dốc đó tại đúng điểm lưới ấy.



Slope field for $\dfrac{dy}{dx} = x - y$.

Ví dụ mẫu · Computing slope field values

For the differential equation $\dfrac{dy}{dx} = x - y$, compute the slope at each of the nine grid points with $x \in \{-1, 0, 1\}$ and $y \in \{-1, 0, 1\}$, and organize the results in a table.

Substitute each pair (x, y) directly into $x - y$:

x	y	$x - y$	slope of segment
-1	-1	0	horizontal
-1	0	-1	falls at 45°
-1	1	-2	steep, falling
0	-1	1	rises at 45°
0	0	0	horizontal
0	1	-1	falls at 45°
1	-1	2	steep, rising
1	0	1	rises at 45°
1	1	0	horizontal

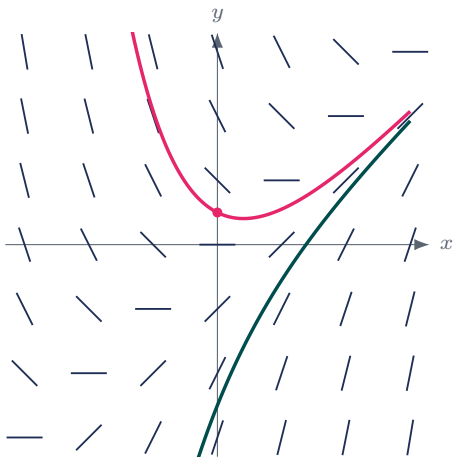
Notice that every point with $y = x$ produces a slope of 0 — this line is where segments are horizontal in the figure above, no matter which of these nine points we started from. This systematic substitution, repeated over a full grid, is exactly how the slope field figure was produced.

7.3.2 Reasoning Using Slope Fields

Two exam-style skills follow directly from being able to read a slope field. First, **matching**: given several differential equations and several slope-field pictures, identify which field belongs to which equation by looking for distinguishing features — where the slope is 0 (the **nullcline**, found by solving $f(x, y) = 0$), where the slope is undefined (usually where a denominator vanishes), whether the pattern depends on x only, on y only, or on both, and any symmetry (for instance, $f(x, y) = f(-x, y)$ gives a field symmetric about the y-axis). Second, **sketching a particular solution curve**: starting from a given point, follow the direction of the nearby segments, moving in small steps in both the increasing-x and decreasing-x directions, always keeping the curve locally tangent to the surrounding segments. A solution curve can never cross itself, and at any point it must be tangent to the segment drawn there.

VN GIẢI THÍCH TIẾNG VIỆT

Khi cho một phương trình vi phân và yêu cầu chọn đúng trường độ dốc tương ứng (trong nhiều lựa chọn), hãy tìm những đặc điểm dễ nhận ra: nơi độ dốc bằng 0 (đường *nullcline*, giải $\frac{dy}{dx} = 0$), nơi độ dốc không xác định (thường là mẫu số bằng 0), và độ dốc có phụ thuộc vào x, vào y, hay cả hai. Để vẽ một nghiệm riêng đi qua một điểm cho trước, xuất phát từ điểm đó và "đi theo" hướng của các đoạn thẳng lân cận, di chuyển từng bước nhỏ theo cả hai chiều x tăng và x giảm.



Slope field for $\frac{dy}{dx} = x - y$ with two particular solution curves traced through the field; the marked point is $(0, 0.5)$.

Ví dụ mẫu · Tracing a solution curve

Using the slope field for $\frac{dy}{dx} = x - y$ shown above, describe the shape of the solution curve through the point $(0, 0.5)$ (the pink curve, marked with a dot).

At $(0, 0.5)$, the slope is $x - y = 0 - 0.5 = -0.5$: negative, so the curve initially heads down and to the right. As x increases and y decreases along the curve, the value $x - y$ increases toward 0 (both changes push it in the same direction), so the curve flattens out as it approaches the line $y = x$ — the nullcline, where every segment in the field is horizontal. Just below that line, $x - y > 0$, so the curve is pulled back upward, and it settles into a gentle rise that continues to track the surrounding segments as $x \to \infty$. This is exactly the qualitative behavior the pink curve in the figure displays: a shallow dip followed by a steady climb, staying close to the direction of the field at every point it passes through.

(!) Lỗi thường gặp: When matching a differential equation to a slope field, do not just look at one or two segments — check whether the *pattern as a whole* matches. A field for $\frac{dy}{dx} = x - y$ and one for $\frac{dy}{dx} = x + y$ can look similar near the origin (both have slope 0 at $(0, 0)$) but differ sharply elsewhere: the first has slope 0 along the whole line $y = x$, the second along the whole line $y = -x$. Always identify the full nullcline, not just a single matching point.

The table below organizes this reasoning for four common differential equations, to make the matching process concrete.

Equation	Nullcline (slope $= 0$)	Notable feature
$\frac{dy}{dx} = x$	the line $x = 0$	slope depends only on x; every point on a given vertical line has the same slope
$\frac{dy}{dx} = y$	the line $y = 0$	slope depends only on y; every point on a given horizontal line has the same slope
$\frac{dy}{dx} = xy$	both axes, $x = 0$ or $y = 0$	180° rotational symmetry about the origin, since $f(-x, -y) = (-x)(-y) = xy = f(x, y)$
$\frac{dy}{dx} = x^2 + y^2$	only the single point $(0, 0)$	slope is never negative (a sum of squares), so every solution curve is non-decreasing

7.4 Approximating Solutions Using Euler's Method

Slope fields show the *shape* of solutions without solving the equation; **Euler's Method** goes one step further and produces *numerical estimates* of a particular solution, again without ever solving the differential equation symbolically. The idea is to start at the known point (x_0, y_0) and repeatedly step forward along the tangent line direction given by the differential equation itself.

Euler's Method: given $\frac{dy}{dx} = f(x, y)$, a starting point (x_0, y_0), and a step size Δx, generate successive approximations by

$$x_{n+1} = x_n + \Delta x, \qquad y_{n+1} = y_n + f(x_n, y_n) \cdot \Delta x.$$

At each step, the slope $f(x_n, y_n)$ is evaluated using the *current* point (x_n, y_n) — never the point being predicted. Smaller Δx gives a more accurate approximation but requires more steps.

VN **GIẢI THÍCH TIẾNG VIỆT**

Phương pháp Euler xấp xỉ nghiệm của một phương trình vi phân bằng cách đi theo đường tiếp tuyến từng bước nhỏ: tại mỗi bước, ta dùng độ dốc hiện tại $f(x_n, y_n) = \frac{dy}{dx}$ tại điểm đang đứng để dự đoán điểm tiếp theo, theo công thức $y_{n+1} = y_n + f(x_n, y_n) \cdot \Delta x$. Bước nhảy Δx càng nhỏ, xấp xỉ càng chính xác – nhưng với Δx lớn, sai số có thể đáng kể, đặc biệt khi đường cong nghiệm thật cong nhiều (độ lõm/lồi lớn).

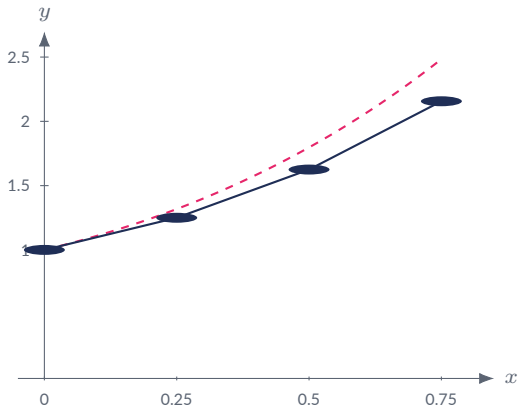
Ví dụ mẫu · Euler's Method by hand

Use Euler's Method with step size $\Delta x = 0.25$ to approximate $y(0.75)$ for $\frac{dy}{dx} = x + y$, given $y(0) = 1$. Show three steps in a table.

Here $f(x, y) = x + y$. Apply $y_{n+1} = y_n + f(x_n, y_n)(0.25)$ three times:

n	x_n	y_n	$f(x_n, y_n) = x_n + y_n$	$y_{n+1} = y_n + f \cdot (0.25)$
0	0	1	1	$1 + 1(0.25) = 1.25$
1	0.25	1.25	1.5	$1.25 + 1.5(0.25) = 1.625$
2	0.5	1.625	2.125	$1.625 + 2.125(0.25) = 2.15625$

So $y(0.75) \approx 2.156$. For comparison, this equation can actually be solved exactly (it is linear, using a technique beyond this unit): the exact solution is $y = 2e^x - x - 1$, giving $y(0.75) = 2e^{0.75} - 1.75 \approx 2.484$. The Euler estimate ($2.156$) is noticeably below the exact value (2.484) because $y'' = 2e^x > 0$ everywhere — the true curve is concave up, so each tangent-line step lies *below* the curve it is approximating, and the error accumulates in the same direction at every step.



Euler's Method (solid polyline, $\Delta x = 0.25$) approximating the exact solution (dashed) of $\frac{dy}{dx} = x + y$, $y(0) = 1$. Because the exact solution is concave up, every tangent-line step falls below the true curve.

(!) Lỗi thường gặp: The single most common Euler's Method error is evaluating the slope f at the *new* point (x_{n+1}, y_{n+1}) instead of the *current* point (x_n, y_n) — but y_{n+1} is not known yet until after that slope is used. Always compute $f(x_n, y_n)$ first, using values already in hand, and only then step forward. A second common error is adding the raw slope $f(x_n, y_n)$ instead of $f(x_n, y_n) \cdot \Delta x$ — forgetting to scale the slope by the step size.

Reducing Δx always brings the Euler approximation closer to the exact value, since more (smaller) tangent-line steps track the true curve more faithfully. The table below repeats the estimate of $y(0.75)$ for $\dfrac{dy}{dx} = x + y$, $y(0) = 1$, from the worked example above, using progressively smaller step sizes (and correspondingly more steps) to reach the same target $x = 0.75$:

Δx	Number of steps	Euler estimate of $y(0.75)$	Exact value
0.75	1	1.75	2.484
0.375	2	2.03125	2.484
0.25	3	2.15625	2.484

As Δx shrinks from 0.75 to 0.25, the Euler estimate climbs steadily from 1.75 toward the exact value 2.484 — exactly the behavior expected of a convergent numerical method, though full convergence to three decimal places would require far smaller steps than are practical to compute by hand.

7.5 Finding General Solutions Using Separation of Variables

Many first-order differential equations that arise from the modeling situations in Section 7.1 have the special form $\frac{dy}{dx} = g(x)h(y)$ — the right-hand side factors into a piece depending only on x times a piece depending only on y. Such equations are called **separable**, and they can be solved completely using algebra and integration.

Separation of variables: to solve $\frac{dy}{dx} = g(x)h(y)$, divide both sides by $h(y)$ (assuming $h(y) \neq 0$) and multiply by dx to separate the variables:

$$\frac{dy}{h(y)} = g(x)\,dx,$$

then integrate each side independently:

$$\int \frac{dy}{h(y)} = \int g(x)\,dx.$$

Each integration introduces its own constant of integration; combine them into a single constant C on one side. The result is the **general solution** — a family of functions, one for every value of C.

VN **GIẢI THÍCH TIẾNG VIỆT**

Tách biến (separation of variables) áp dụng cho phương trình có dạng $\frac{dy}{dx} = g(x)h(y)$: ta đưa tất cả các biểu thức chứa y (kể cả dy) sang một vế, tất cả biểu thức chứa x (kể cả dx) sang vế còn lại – $\frac{dy}{h(y)} = g(x)\,dx$ – rồi lấy nguyên hàm hai vế **độc lập**. Mỗi lần lấy nguyên hàm sinh ra một hằng số cộng; gộp hai hằng số đó lại thành một hằng số C duy nhất ở nghiệm tổng quát.

Ví dụ mẫu · Separable equation, exponential form

Find the general solution of $\frac{dy}{dx} = x^2 y$.

Separate (assuming $y \neq 0$):

$$\frac{dy}{y} = x^2\,dx \implies \int \frac{dy}{y} = \int x^2\,dx \implies \ln|y| = \frac{x^3}{3} + C.$$

Exponentiate both sides: $|y| = e^{x^3/3 + C} = e^C e^{x^3/3}$, so $y = \pm e^C e^{x^3/3}$. Writing $A = \pm e^C$ for the arbitrary nonzero constant (and noting $y = 0$ is also a solution, absorbed into $A = 0$):

$$y = Ae^{x^3/3}.$$

Check: $\frac{dy}{dx} = A \cdot x^2 e^{x^3/3} = x^2\left(Ae^{x^3/3}\right) = x^2 y$. ✓

Ví dụ mẫu · Separable equation, power form

Find the general solution of $\frac{dy}{dx} = \frac{x+1}{y^2}$.

Separate:

$$y^2\,dy = (x+1)\,dx \implies \int y^2\,dy = \int (x+1)\,dx \implies \frac{y^3}{3} = \frac{x^2}{2} + x + C.$$

Multiply by 3 and rename the constant: $y^3 = \dfrac{3}{2}x^2 + 3x + K$, so

$$y = \left(\frac{3}{2}x^2 + 3x + K\right)^{1/3}.$$

Check: let $u = \frac{3}{2}x^2 + 3x + K$, so $y = u^{1/3}$ and $\dfrac{dy}{dx} = \frac{1}{3}u^{-2/3}\cdot(3x+3) = (x+1)u^{-2/3}$. Meanwhile $\dfrac{x+1}{y^2} = \dfrac{x+1}{u^{2/3}} = (x+1)u^{-2/3}$ — the two match. ✓

Ví dụ mẫu · Separable equation, logarithmic form

Find the general solution of $\dfrac{dy}{dx} = xe^{-y}$.

Separate:

$$e^y\,dy = x\,dx \implies \int e^y\,dy = \int x\,dx \implies e^y = \frac{x^2}{2} + C.$$

Solve for y by taking a logarithm of both sides:

$$y = \ln\left(\frac{x^2}{2} + C\right),$$

valid wherever $\frac{x^2}{2} + C > 0$.

Check: $\dfrac{dy}{dx} = \dfrac{x}{x^2/2 + C}$, while $xe^{-y} = x\cdot e^{-\ln(x^2/2+C)} = \dfrac{x}{x^2/2+C}$ — matches. ✓

(!) Lỗi thường gặp: When separating $\dfrac{dy}{dx} = g(x)h(y)$, every term must carry its correct sign to the correct side. A frequent mistake: rewriting $\dfrac{dy}{dx} = -\dfrac{x}{y}$ as $y\,dy = x\,dx$ (dropping the minus sign) instead of the correct $y\,dy = -x\,dx$. The dropped sign turns a family of *circles* ($x^2 + y^2 = K$) into a family of *hyperbolas* ($y^2 - x^2 = K$) — an entirely different geometric answer, not just a numerically wrong constant.

(!) Lỗi thường gặp: Because $\displaystyle\int \frac{dy}{y} = \ln|y| + C$, the correct intermediate step is $\ln|y| = \cdots$, never $\ln y = \cdots$ (which silently assumes $y > 0$ for every x under consideration). Dropping the absolute value can produce a solution with the wrong sign or an artificially restricted domain. The safe habit: always write $|y| = e^{\cdots}$, then convert to $y = \pm e^{\cdots} = Ae^{\cdots}$, where A ranges over all nonzero constants (and sometimes $A = 0$ also works, recovering the constant solution $y = 0$).

7.6 Finding Particular Solutions Using Initial Conditions

Once a general solution is found, an **initial condition** $y(x_0) = y_0$ pins down the value of the arbitrary constant, producing a single **particular solution**. Sometimes, after integrating, the equation relating x and y can be solved cleanly for y — this is an **explicit** solution. Other times the equation cannot (or need not) be rearranged into the form $y = \cdots$, most often because y appears raised to a power of 3 or higher, or in a combination of terms that cannot be isolated; the answer is then left as an **implicit** equation relating x and y, which is still a complete and valid solution.

A subtlety arises whenever solving for y requires taking a square root: algebra alone gives *two* possible branches, $y = +\sqrt{\cdots}$ and $y = -\sqrt{\cdots}$, but only one of them can pass through the given initial point. The initial condition itself decides which branch is correct — if $y_0 > 0$, the positive branch is required; if $y_0 < 0$, the negative branch is required. Once the correct branch is chosen, the domain of the particular solution is also restricted to the interval around x_0 on which the expression under the square root stays non-negative.

VN **GIẢI THÍCH TIẾNG VIỆT**

Sau khi có nghiệm tổng quát (chứa hằng số C), ta thay toạ độ của **điều kiện đầu** (initial condition) $y(x_0) = y_0$ vào để giải ra giá trị cụ thể của C, thu được **nghiệm riêng** (particular solution). Đôi khi, sau khi lấy nguyên hàm, phương trình thu được không thể (hoặc không cần) giải tường minh cho y theo x – ta để nguyên dạng **ẩn (implicit)**, ví dụ $y^3 - y^2 = \frac{x^3}{3}$, và đó vẫn là một câu trả lời hoàn chỉnh, hợp lệ.

Ví dụ mẫu · Particular solution, explicit

Solve the initial value problem $\dfrac{dy}{dx} = 2xy^2$, $y(1) = -1$.

Separate:

$$\frac{dy}{y^2} = 2x\,dx \implies \int y^{-2}\,dy = \int 2x\,dx \implies -\frac{1}{y} = x^2 + C.$$

Apply the initial condition $x = 1$, $y = -1$: $-\dfrac{1}{-1} = 1 + C \Rightarrow 1 = 1 + C \Rightarrow C = 0$. So

$$-\frac{1}{y} = x^2 \implies y = -\frac{1}{x^2}.$$

Check: $\dfrac{dy}{dx} = \dfrac{2}{x^3}$, and $2xy^2 = 2x \cdot \dfrac{1}{x^4} = \dfrac{2}{x^3}$ — matches. Since the solution is undefined at $x = 0$ and the initial condition is at $x = 1$, this particular solution is valid on the interval $x > 0$.

Ví dụ mẫu · Particular solution, implicit

Solve the initial value problem $\dfrac{dy}{dx} = \dfrac{x^2}{y - y^2}$, $y(0) = 2$, leaving the answer in implicit form.

Separate:

$$(y - y^2)\,dy = x^2\,dx \implies \int (y - y^2)\,dy = \int x^2\,dx \implies \frac{y^2}{2} - \frac{y^3}{3} = \frac{x^3}{3} + C.$$

Apply $x = 0$, $y = 2$: $\dfrac{4}{2} - \dfrac{8}{3} = 2 - \dfrac{8}{3} = -\dfrac{2}{3}$, and the right side is $0 + C$, so $C = -\dfrac{2}{3}$. The particular solution is

$$\frac{y^2}{2} - \frac{y^3}{3} = \frac{x^3}{3} - \frac{2}{3}, \qquad \text{or, clearing denominators,} \qquad 3y^2 - 2y^3 = 2x^3 - 4.$$

This cubic in y cannot practically be solved for y in terms of x, so the implicit form is the final answer. **Check** by implicit differentiation: differentiating $3y^2 - 2y^3 = 2x^3 - 4$ gives $6yy' - 6y^2y' = 6x^2$, so $y'(6y - 6y^2) = 6x^2$, and dividing by 6: $y' = \dfrac{x^2}{y - y^2}$ — exactly the original differential equation. ✓ Also, $(0, 2)$ satisfies the implicit equation: $3(4) - 2(8) = 12 - 16 = -4$, and $2(0) - 4 = -4$. ✓

(!) Lỗi thường gặp: Solve first, *then* apply the initial condition — never the reverse. A common error is to integrate one side, immediately substitute the initial values, and only afterward integrate the other side; this loses track of the constant of integration and typically makes it impossible to determine C correctly (or silently sets $C = 0$ without justification). Always finish separating and integrating *both* sides completely, producing exactly one combined constant C in the general solution, before touching the initial condition at all.

7.7 Exponential Models with Differential Equations

7.7.1 The Exponential Growth and Decay Model

The differential equation $\dfrac{dy}{dt} = ky$ — "the rate of change of y is proportional to y itself" — is separable and appears throughout the sciences. Separating variables:

$$\frac{dy}{y} = k\,dt \implies \ln|y| = kt + C \implies y = Ae^{kt}.$$

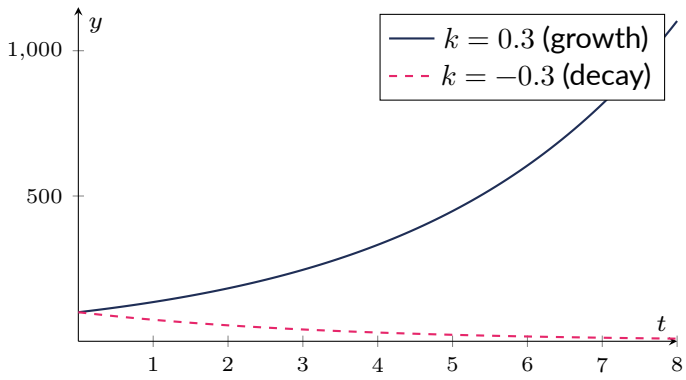
Applying the initial condition $y(0) = y_0$ gives $A = y_0$, so the particular solution is always

$$y = y_0 e^{kt}.$$

Exponential model: $\dfrac{dy}{dt} = ky \iff y = y_0 e^{kt}$. If $k > 0$, y grows without bound (population growth, compound interest); if $k < 0$, y decays toward 0 (radioactive decay). The **half-life** of a decaying quantity is the fixed time $t_{1/2}$ required for y to fall to half its value, independent of the starting amount; it satisfies $t_{1/2} = \dfrac{\ln 2}{|k|}$.

VN **GIẢI THÍCH TIẾNG VIỆT**

Phương trình $\frac{dy}{dt} = ky$ (tốc độ thay đổi tỉ lệ thuận với chính đại lượng đó) luôn có nghiệm $y = y_0 e^{kt}$, trong đó y_0 là giá trị tại $t = 0$. Nếu $k > 0$, đại lượng **tăng trưởng mũ** (ví dụ số lượng vi khuẩn); nếu $k < 0$, đại lượng **phân rã mũ** (ví dụ khối lượng chất phóng xạ). **Bán chu kỳ** (half-life) là khoảng thời gian để đại lượng giảm còn một nửa – không phụ thuộc vào giá trị ban đầu.



Solutions of $\frac{dy}{dt} = ky$ with $y_0 = 100$: exponential growth ($k = 0.3 > 0$) versus exponential decay ($k = -0.3 < 0$).

Ví dụ mẫu · Exponential growth and half-life

(a) A bacterial culture grows according to $\dfrac{dP}{dt} = kP$. At $t = 0$ there are 500 bacteria; after 3 hours there are 1400. Find k, then predict the population after 8 hours.

$P(t) = 500e^{kt}$. Using $P(3) = 1400$: $1400 = 500e^{3k} \Rightarrow e^{3k} = 2.8 \Rightarrow k = \dfrac{\ln 2.8}{3} \approx \dfrac{1.02962}{3} \approx 0.3432$ per hour.

$$P(8) = 500e^{8k} = 500\,(2.8)^{8/3} \approx 500(15.575) \approx 7787 \text{ bacteria.}$$

(b) A radioactive isotope has a half-life of 12 days. Starting with 80 mg, how much remains after 30 days?

Rather than solving for k numerically, use the half-life directly: 30 days is $30/12 = 2.5$ half-lives, so the remaining fraction is $(1/2)^{2.5}$:

$$y(30) = 80\left(\frac{1}{2}\right)^{2.5} = \frac{80}{2^{2.5}} = \frac{80}{4\sqrt{2}} = \frac{20}{\sqrt{2}} = 10\sqrt{2} \approx 14.14 \text{ mg.}$$

(Equivalently, $k = -\dfrac{\ln 2}{12} \approx -0.05776$ per day, and $y(30) = 80e^{30k} = 80e^{-2.5\ln 2} = 80 \cdot 2^{-2.5}$, the same result.)

7.7.2 Newton's Law of Cooling

A related model describes how the temperature of an object changes as it approaches the temperature of its surroundings. **Newton's Law of Cooling** states that the rate of change of an object's temperature T is proportional to the difference between T and the ambient (surrounding) temperature T_{amb}:

$$\frac{dT}{dt} = k(T - T_{amb}).$$

This is separable, with $g(t) = k$ and $h(T) = T - T_{amb}$:

$$\frac{dT}{T - T_{amb}} = k\,dt \implies \ln|T - T_{amb}| = kt + C \implies T - T_{amb} = Ae^{kt} \implies T = T_{amb} + Ae^{kt},$$

where $A = T_0 - T_{amb}$ is the initial temperature difference. Because T must approach T_{amb} as $t \to \infty$ (an object left long enough always settles to room temperature), the term Ae^{kt} must shrink to 0, which forces $k < 0$ — **always**, whether the object is cooling down toward T_{amb} or warming up toward it. A positive k would make T diverge away from T_{amb} without bound, which does not describe cooling or warming toward equilibrium at all.

VN GIẢI THÍCH TIẾNG VIỆT

Định luật làm nguội Newton mô tả tốc độ thay đổi nhiệt độ của một vật tỉ lệ với **hiệu** giữa nhiệt độ của vật và nhiệt độ môi trường xung quanh: $\frac{dT}{dt} = k(T - T_{amb})$. Đây là một dạng đặc biệt của phương trình tách biến, với nghiệm $T = T_{amb} + Ae^{kt}$ – nhiệt độ luôn tiến dần về T_{amb} khi $t \to \infty$, vì vậy hằng số k trong quy ước này luôn **âm**, bất kể vật đang nguội đi hay đang ấm lên.

Ví dụ mẫu · Newton's Law of Cooling

A cup of coffee at $90°$C is placed in a room kept at $20°$C. After 10 minutes, the coffee has cooled to $60°$C. **(a)** Find k. **(b)** Find the coffee's temperature after 20 minutes.

(a) Here $T_{amb} = 20$ and $A = T_0 - T_{amb} = 90 - 20 = 70$, so $T(t) = 20 + 70e^{kt}$. Using $T(10) = 60$:

$$60 = 20 + 70e^{10k} \implies 40 = 70e^{10k} \implies e^{10k} = \frac{4}{7} \implies k = \frac{\ln(4/7)}{10} \approx -0.05596 \text{ per minute.}$$

As expected, $k < 0$. **(b)** Since $20 = 2 \times 10$, we can reuse $e^{10k} = 4/7$ directly: $e^{20k} = \left(e^{10k}\right)^2 = \left(\frac{4}{7}\right)^2 = \frac{16}{49}$.

$$T(20) = 20 + 70 \cdot \frac{16}{49} = 20 + \frac{1120}{49} = 20 + 22.857 \approx 42.9°\text{C}.$$

As $t \to \infty$, $T \to 20°$C — the coffee approaches, but never exactly reaches, room temperature.

(!) Lỗi thường gặp: Two mix-ups are common with Newton's Law of Cooling, $\frac{dT}{dt} = k(T - T_{amb})$: (1) forgetting the ambient term entirely and writing $\frac{dT}{dt} = kT$, which models unrestricted exponential growth or decay toward *zero* rather than toward room temperature; (2) assuming $k > 0$ describes cooling. In this convention k is **always negative**, whether the object is losing heat toward T_{amb} or gaining heat toward T_{amb} — only the sign of the initial difference $A = T_0 - T_{amb}$ changes between the two cases, never the sign of k.

7.8 The Logistic Differential Equation

The exponential model $\frac{dy}{dt} = ky$ predicts growth (or decay) without limit, which is unrealistic for most real populations: food, space, and resources eventually cap how large a population can get. The **logistic differential equation** builds in such a limit:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right),$$

where L is the **carrying capacity** — the maximum population the environment can sustain in the long run — and $k > 0$ is a growth-rate constant. When P is small compared to L, the factor $\left(1 - \frac{P}{L}\right)$ is close to 1, so $\frac{dP}{dt} \approx kP$ and the population grows nearly exponentially, just as in the unrestricted model. As P approaches L, the factor $\left(1 - \frac{P}{L}\right)$ shrinks toward 0, so growth slows and eventually stops.

Ví dụ mẫu · Logistic equilibria and max growth

For $\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$: **(a)** verify that $P = 0$ and $P = L$ are both equilibrium (constant) solutions; **(b)** find the population level at which the growth rate $\frac{dP}{dt}$ is largest, and its value there.

(a) A constant function $P = c$ is an equilibrium solution exactly when $\frac{dP}{dt} = 0$ at that value. At $P = 0$: $kP(1 - P/L) = k(0)(1 - 0) = 0$. ✓ At $P = L$: $kL(1 - L/L) = kL(1 - 1) = kL(0) = 0$. ✓ Both are equilibrium solutions.

(b) Write $g(P) = kP\left(1 - \frac{P}{L}\right) = kP - \frac{k}{L}P^2$ — a downward-opening parabola in P (since $k > 0$). Its maximum occurs where $g'(P) = 0$:

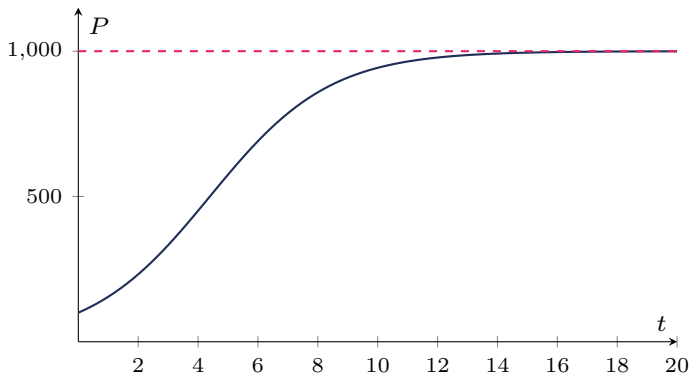
$$g'(P) = k - \frac{2k}{L}P = 0 \implies P = \frac{L}{2}.$$

The maximum growth rate is $g\left(\frac{L}{2}\right) = k \cdot \frac{L}{2}\left(1 - \frac{1}{2}\right) = k \cdot \frac{L}{2} \cdot \frac{1}{2} = \frac{kL}{4}$. So the population grows fastest exactly halfway to capacity, at $P = L/2$, and that fastest rate is $\frac{kL}{4}$.

Logistic behavior: $P = 0$ and $P = L$ are equilibrium solutions; for $0 < P < L$, $\frac{dP}{dt} > 0$ and the population increases toward L; for $P > L$, $\frac{dP}{dt} < 0$ and the population decreases back toward L (the population can never cross L, only approach it). Growth is fastest at $P = L/2$, giving the graph of P versus t its characteristic **S-shape**: nearly exponential at first, an inflection point at $P = L/2$, then leveling off toward L. It can be shown that the closed-form solution has the shape $P(t) = \frac{L}{1 + Ae^{-kt}}$ for a constant A determined by the initial population.

VN **GIẢI THÍCH TIẾNG VIỆT**

Phương trình vi phân logistic $\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$ mô tả tăng trưởng có giới hạn: hằng số L gọi là **sức chứa môi trường** (carrying capacity) – giá trị lớn nhất mà quần thể có thể đạt tới lâu dài. Khi P còn nhỏ so với L, phương trình gần giống $\frac{dP}{dt} \approx kP$ (tăng gần như hàm mũ); khi P tiến gần L, tốc độ tăng trưởng chậm dần về 0. Tốc độ tăng trưởng đạt **lớn nhất** khi $P = \frac{L}{2}$, tạo nên hình dạng đường cong chữ **S** đặc trưng.



Logistic growth $\frac{dP}{dt} = kP(1 - P/L)$ with $L = 1000$, $k = 0.5$: the S-shaped curve rises fastest near $P = L/2 = 500$ (around $t \approx 4.4$) and levels off toward the carrying capacity $L = 1000$ (dashed).

Exponential versus logistic growth: both models start out looking similar for small P, but they describe very different long-term behavior.

	Exponential $\frac{dy}{dt} = ky$	**Logistic** $\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$
Equilibrium solution(s)	only $y = 0$	$P = 0$ and $P = L$
Long-term behavior	unbounded growth (or decay to 0)	levels off at the carrying capacity L
Shape of graph	purely convex (for $k > 0$)	S-shaped: convex, then concave, with an inflection at $P = L/2$

Summary of solution techniques in this unit:

Technique	**What it produces**	**Requires the equation to be separable?**
Slope field	a picture of the direction of every solution curve; used to sketch or match shapes	no — works for any $\frac{dy}{dx} = f(x, y)$
Euler's Method	a numerical table of estimated y-values at chosen x-values	no — works for any $\frac{dy}{dx} = f(x, y)$
Separation of variables	an exact formula for y (explicit or implicit), the general or a particular solution	yes — only when $f(x, y) = g(x)h(y)$

Slope fields and Euler's Method are the fallback tools when an equation is not separable (such as $\frac{dy}{dx} = x - y$ or $\frac{dy}{dx} = x + y$, used throughout Sections 7.3–7.5); separation of variables is reserved for equations with the special product structure, but when it applies it gives an exact answer rather than an approximation.

7.9 Euler's Method: Tables, Concavity, and Error

Phần dành riêng cho AP Calculus BC: trong CED, phương pháp Euler (Topic 7.5) và mô hình logistic (Topic 7.9) chỉ thuộc đề BC. Phần này trình bày dạng câu hỏi tự luận (FRQ) của hai chủ đề: Euler nhiều bước trên bảng, lập luận ước lượng thừa/thiếu bằng độ lõm $\frac{d^2y}{dx^2}$, và giải tường minh phương trình logistic bằng tách phân thức.

The earlier section on Euler's Method introduced the recursion $y_{n+1} = y_n + f(x_n, y_n)\,\Delta x$. On the BC exam, Euler's Method almost never appears alone: it is paired with a second-derivative computation and a request to decide whether the estimate is too large or too small. This section develops that full chain of reasoning.

7.9.1 The second derivative from a differential equation

If $\frac{dy}{dx} = f(x, y)$, then $\frac{d^2y}{dx^2}$ is found by differentiating $f(x, y)$ *implicitly* with respect to x — remembering that y is a function of x — and then replacing every $\frac{dy}{dx}$ by $f(x, y)$. The result is a formula in x and y alone, whose sign at any point tells the concavity of the solution curve through that point.

Concavity decides over- or underestimate

Let $y = f(x)$ be a solution curve.

- If $\dfrac{d^2y}{dx^2} > 0$ (concave up) on the interval in question, tangent lines lie *below* the curve, so a tangent-line approximation — and an Euler approximation built from tangent segments — is an **underestimate**.
- If $\dfrac{d^2y}{dx^2} < 0$ (concave down) on the interval, tangent lines lie *above* the curve, and the approximation is an **overestimate**.

For a single tangent-line approximation, the sign of $\dfrac{d^2y}{dx^2}$ near the point of tangency suffices. For a multi-step Euler approximation, the concavity must hold along the *entire* path from the initial point to the target.

VN **GIẢI THÍCH TIẾNG VIỆT**

Để biết ước lượng bằng tiếp tuyến hay bằng Euler là **thừa** hay **thiếu**, ta tính $\frac{d^2y}{dx^2}$ bằng cách đạo hàm ẩn biểu thức $\frac{dy}{dx} = f(x,y)$ theo x (nhớ y là hàm của x), rồi **thay** $\frac{dy}{dx}$ **bằng** $f(x,y)$. Nếu $\frac{d^2y}{dx^2} > 0$ (lõm lên) thì tiếp tuyến nằm dưới đồ thị nên ước lượng **thiếu** (underestimate); nếu $\frac{d^2y}{dx^2} < 0$ (lõm xuống) thì ước lượng **thừa** (overestimate). Khi trả lời FRQ, phải nêu rõ *dấu* của $\frac{d^2y}{dx^2}$ và *trên khoảng nào* — chỉ viết "concave up" mà không có lý do sẽ không được điểm.

Ví dụ mẫu · A complete FRQ chain

Let $y = f(x)$ be the particular solution to $\dfrac{dy}{dx} = 2x - y$ with $f(0) = 1$.
(a) Write an equation of the line tangent to the graph of f at $x = 0$, and use it to approximate $f(0.2)$.
(b) Find $\dfrac{d^2y}{dx^2}$ in terms of x and y. Is the approximation in (a) an overestimate or an underestimate? Justify.
(c) Use Euler's Method, starting at $x = 0$ with two steps of equal size, to approximate $f(1)$.

(a) At $(0,1)$: $\dfrac{dy}{dx} = 2(0) - 1 = -1$. Tangent line: $y = 1 - x$. So $f(0.2) \approx 1 - 0.2 = 0.8$.

(b) Differentiate $\dfrac{dy}{dx} = 2x - y$ with respect to x:

$$\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx} = 2 - (2x - y) = 2 - 2x + y.$$

At $(0,1)$, $\dfrac{d^2y}{dx^2} = 2 - 0 + 1 = 3 > 0$, so the graph of f is concave up near $x = 0$ and the tangent line lies below the curve there: 0.8 is an **underestimate** of $f(0.2)$.

(c) $\Delta x = 0.5$.

x_n	y_n	slope $2x_n - y_n$	$y_{n+1} = y_n + (\text{slope})(0.5)$
0	1	-1	$1 + (-1)(0.5) = 0.5$
0.5	0.5	0.5	$0.5 + (0.5)(0.5) = 0.75$

So $f(1) \approx 0.75$. (The exact solution is $f(x) = 2x - 2 + 3e^{-x}$, with $f''(x) = 3e^{-x} > 0$ for every x; the curve is concave up everywhere, and indeed $f(1) = 3/e \approx 1.104 > 0.75$.)

7.9.2 A concave-down example and the effect of step size

Ví dụ mẫu · Overestimate from concavity

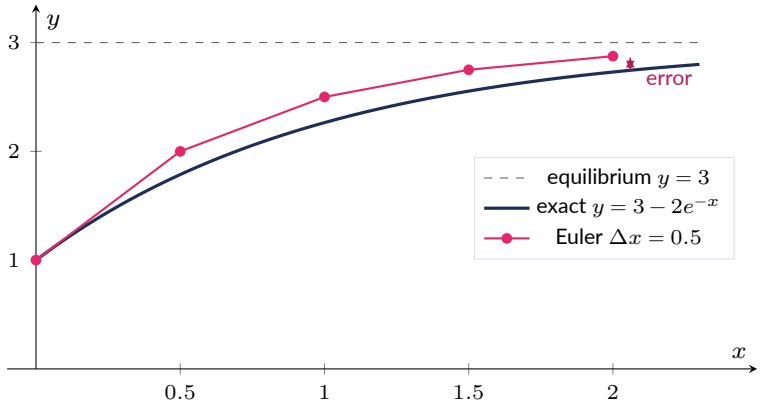
Let $y = f(x)$ be the solution to $\dfrac{dy}{dx} = 3 - y$ with $f(0) = 1$.
(a) Use Euler's Method with step size 0.5 to approximate $f(2)$.
(b) Show that every solution curve with $y < 3$ is concave down, and explain why the estimate in (a) is an overestimate.

(a) Four steps of size 0.5:

n	x_n	y_n	slope $3 - y_n$	y_{n+1}
0	0	1	2	$1 + 2(0.5) = 2$
1	0.5	2	1	$2 + 1(0.5) = 2.5$
2	1	2.5	0.5	$2.5 + 0.5(0.5) = 2.75$
3	1.5	2.75	0.25	$2.75 + 0.25(0.5) = 2.875$

So $f(2) \approx 2.875$.

(b) $\dfrac{d^2y}{dx^2} = -\dfrac{dy}{dx} = -(3 - y) = y - 3$, which is negative whenever $y < 3$. The solution starts at $y = 1 < 3$ and, since $\dfrac{dy}{dx} = 3 - y > 0$ while $y < 3$, it increases toward the equilibrium $y = 3$ but never reaches it. So $y < 3$ and $\dfrac{d^2y}{dx^2} < 0$ on all of $[0, 2]$: the curve is concave down, each Euler segment lies above it, and 2.875 is an **overestimate**. (Exact: $f(x) = 3 - 2e^{-x}$, $f(2) \approx 2.729$.)



Euler's Method on a concave-down solution: every segment starts on (or above) the true curve and follows a tangent direction that rises faster than the curve, so the polyline stays above the exact solution and the error grows with each step.

Halving the step size roughly halves the error — Euler's Method is a *first-order* method. For the same problem:

Step size Δx	1	0.5	0.25	0.1
Number of steps	2	4	8	20
Euler estimate of $f(2)$	3	2.875	2.800	2.757
Error (exact 2.729)	0.271	0.146	0.071	0.028

7.9.3 Stepping backward and stepping from a table

Euler's Method works equally well with a *negative* step: to approximate $f(0.8)$ from a known value $f(1)$, use $\Delta x = -0.1$. The formula is unchanged; only the sign of Δx differs. Concavity reasoning

is also unchanged: a concave-up curve lies above all its tangent lines on *both* sides of the point of tangency, so the estimate is still an underestimate.

Ví dụ mẫu · Negative step size

Let $y = f(x)$ solve $\dfrac{dy}{dx} = xy$ with $f(1) = 2$. Use Euler's Method with two steps of size $\Delta x = -0.1$ to approximate $f(0.8)$.

Step 1: slope at $(1, 2)$ is $1 \cdot 2 = 2$; $y_1 = 2 + 2(-0.1) = 1.8$ at $x = 0.9$.
Step 2: slope at $(0.9, 1.8)$ is $0.9 \cdot 1.8 = 1.62$; $y_2 = 1.8 + 1.62(-0.1) = 1.638$ at $x = 0.8$.

So $f(0.8) \approx 1.638$. Here $\dfrac{d^2y}{dx^2} = y + x\dfrac{dy}{dx} = y + x^2y = y(1+x^2) > 0$ while $y > 0$, so the estimate is an underestimate. (Exact: $f(x) = 2e^{(x^2-1)/2}$, $f(0.8) = 2e^{-0.18} \approx 1.671$.)

(!) Lỗi thường gặp: For a multi-step Euler estimate, the sign of $\dfrac{d^2y}{dx^2}$ at the *starting point alone* does not justify "over" or "under". A complete justification shows the sign holds along the whole path — for instance by showing y stays in a region where $\frac{d^2y}{dx^2}$ keeps one sign, as in the $\frac{dy}{dx} = 3 - y$ example. When the second derivative changes sign along the way, no conclusion can be drawn from concavity.

7.10 Logistic Models: Long-Run Behavior and Explicit Solutions

7.10.1 Recognizing a logistic equation

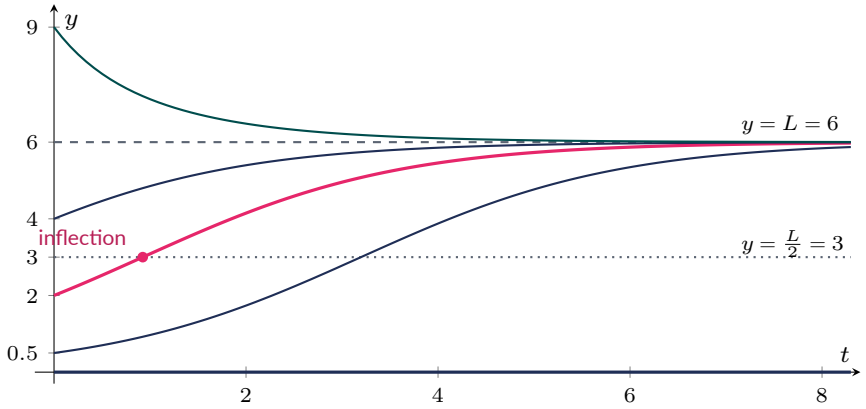
A logistic differential equation can arrive in several algebraically equivalent forms. All of them are a constant times y times (a constant minus y):

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right) \quad \Longleftrightarrow \quad \frac{dy}{dt} = \frac{k}{L}\,y\,(L - y) \quad \Longleftrightarrow \quad \frac{dy}{dt} = ky - \frac{k}{L}y^2.$$

In the form $\dfrac{dy}{dt} = c\,y\,(a - y)$ with $c > 0$, the carrying capacity is $L = a$ and the relative growth constant is $k = ca$. In the expanded form $\dfrac{dy}{dt} = \alpha y - \beta y^2$, factor out αy: $\alpha y(1 - \frac{\beta}{\alpha}y)$, so $L = \dfrac{\alpha}{\beta}$.

What the logistic equation tells you without solving it (for $\frac{dy}{dt} = ky(1 - \frac{y}{L})$, $k > 0$):

- **Long-run behavior:** if $y(0) > 0$, then $\lim\limits_{t\to\infty} y(t) = L$. If $y(0) = 0$, then $y \equiv 0$.
- **Direction:** $0 < y(0) < L$ gives an increasing solution; $y(0) > L$ gives a decreasing solution; $y(0) = L$ gives the constant solution $y = L$.
- **Fastest growth:** $\frac{dy}{dt}$ is a quadratic in y with maximum at $y = \frac{L}{2}$; there $\frac{dy}{dt} = \frac{kL}{4}$, and the solution curve has its point of inflection.
- **Concavity:** $\dfrac{d^2y}{dt^2} = k\left(1 - \dfrac{2y}{L}\right)\dfrac{dy}{dt}$; for $0 < y < L$ this is positive when $y < \frac{L}{2}$ and negative when $y > \frac{L}{2}$.



Solutions of $\frac{dy}{dt} = \frac{1}{8}y(6-y)$ with $y(0) = 0.5,\ 2,\ 4,\ 9$ (and the equilibrium $y = 0$). Every positive solution tends to $L = 6$; those starting below $L/2 = 3$ have an inflection point when they cross $y = 3$; the one starting above 6 decreases toward 6 and is concave up.

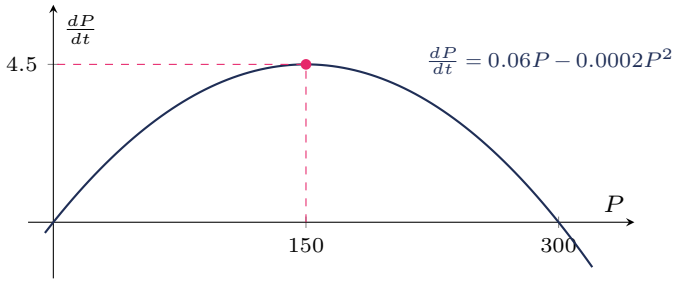
Ví dụ mẫu · Reading an expanded logistic equation

A population satisfies $\dfrac{dP}{dt} = 0.06P - 0.0002P^2$, with $P(0) = 40$.
(a) Find $\lim\limits_{t\to\infty} P(t)$. **(b)** Find the population when it is growing fastest, and that maximum rate.
(c) If instead $P(0) = 400$, find $\lim\limits_{t\to\infty} P(t)$ and describe the behavior.

(a) $0.06P - 0.0002P^2 = 0.06P\left(1 - \dfrac{0.0002}{0.06}P\right) = 0.06P\left(1 - \dfrac{P}{300}\right)$, so $L = 300$ and $k = 0.06$. Since $P(0) = 40 > 0$, $\lim\limits_{t\to\infty} P(t) = 300$.

(b) Fastest growth at $P = \dfrac{L}{2} = 150$, where $\dfrac{dP}{dt} = 0.06(150) - 0.0002(150)^2 = 9 - 4.5 = 4.5$ per unit time (equivalently $\frac{kL}{4} = \frac{0.06\cdot 300}{4}$).

(c) $P(0) = 400 > 300$ gives $\dfrac{dP}{dt} = 0.06(400)(1 - \frac{400}{300}) < 0$: the population decreases, approaching 300 from above, so the limit is still 300.



The growth rate as a function of the population is a downward parabola with zeros at the equilibria $P = 0$ and $P = L = 300$ and its vertex at $P = L/2 = 150$.

7.10.2 Solving the logistic equation explicitly

The logistic equation is separable, and the integral it produces is exactly the kind handled by linear partial fractions (Section 6.12).

General solution of the logistic equation

Separate variables in $\dfrac{dy}{dt} = \dfrac{k}{L}\,y\,(L-y)$:

$$\int \frac{L}{y(L-y)}\,dy = \int k\,dt.$$

Partial fractions: $\dfrac{L}{y(L-y)} = \dfrac{1}{y} + \dfrac{1}{L-y}$ (check: $\frac{(L-y)+y}{y(L-y)}$). Integrating, for $0 < y < L$,

$$\ln y - \ln(L-y) = kt + C \implies \frac{y}{L-y} = Be^{kt} \implies y(t) = \frac{L}{1+Ae^{-kt}}, \qquad A = \frac{L-y(0)}{y(0)}.$$

(The same formula holds for $y(0) > L$, where $A < 0$.)

VN **GIẢI THÍCH TIẾNG VIỆT**

Phương trình logistic là phương trình **tách biến**; khi tách, ta gặp tích phân $\int \frac{dy}{y(L-y)}$ — chính là dạng **tách phân thức** với hai nhân tử bậc nhất phân biệt (y và $L-y$). Kết quả cuối cùng luôn có dạng $y = \dfrac{L}{1+Ae^{-kt}}$ với $A = \dfrac{L-y_0}{y_0}$. Nhờ công thức tường minh, ta trả lời được câu hỏi "**lúc nào** tăng nhanh nhất" (giải $y(t) = \frac{L}{2}$ được $t = \frac{\ln A}{k}$), còn câu hỏi "tăng nhanh nhất **khi** y **bằng bao nhiêu**" thì không cần giải: luôn là $y = \frac{L}{2}$.

Ví dụ mẫu · BC free-response, logistic

The number of people $y(t)$ (in hundreds) who have heard a news story t days after it breaks satisfies $\dfrac{dy}{dt} = \dfrac{1}{8}\,y\,(6-y)$, with $y(0) = 2$.
(a) Find $\lim\limits_{t\to\infty} y(t)$. If instead $y(0) = 8$, what is $\lim\limits_{t\to\infty} y(t)$?
(b) For what value of y is y increasing most rapidly? Justify.
(c) Find $y(t)$ by separation of variables, and find the time at which y is increasing most rapidly.

(a) $\dfrac{dy}{dt} = \dfrac{1}{8}y(6-y)$ is logistic with $L = 6$. Since $0 < y(0) = 2 < 6$, $\lim\limits_{t\to\infty} y(t) = 6$ (600 people). If $y(0) = 8 > 6$, y decreases toward 6, and the limit is again 6.

(b) $\dfrac{dy}{dt} = \dfrac{1}{8}(6y - y^2)$ is a quadratic in y with maximum where $\dfrac{d}{dy}\left[\frac{1}{8}(6y-y^2)\right] = \frac{1}{8}(6-2y) = 0$, i.e. $y = 3$. Equivalently, $\dfrac{d^2y}{dt^2} = \dfrac{1}{8}(6-2y)\dfrac{dy}{dt}$ changes from positive to negative as y passes 3 (with $\frac{dy}{dt} > 0$ for $0 < y < 6$), so $\frac{dy}{dt}$ has its maximum there.

(c) Separate: $\displaystyle\int \frac{dy}{y(6-y)} = \int \frac{1}{8}\,dt$. Partial fractions: $\dfrac{1}{y(6-y)} = \dfrac{1}{6}\left(\dfrac{1}{y} + \dfrac{1}{6-y}\right)$. So

$$\frac{1}{6}(\ln y - \ln(6-y)) = \frac{t}{8} + C \implies \ln\frac{y}{6-y} = \frac{3}{4}t + C_1 \implies \frac{y}{6-y} = Be^{3t/4}.$$

At $t = 0$: $\frac{2}{4} = B$, so $B = \frac{1}{2}$. Solving $y = \frac{1}{2}e^{3t/4}(6-y)$ for y:

$$y(t) = \frac{6}{1+2e^{-3t/4}}.$$

Fastest growth when $y = 3$: $1 + 2e^{-3t/4} = 2 \Rightarrow e^{-3t/4} = \frac{1}{2} \Rightarrow t = \frac{4}{3}\ln 2 \approx 0.924$ days.

(!) Lỗi thường gặp: Two different questions sound alike: "for what value of y is y growing fastest?" (answer $y = \frac{L}{2}$, no solving needed) and "at what *time* is y growing fastest?" (requires the explicit solution). Also, the carrying capacity of $\frac{dy}{dt} = c\,y\,(a - y)$ is a, not ca; for $\frac{dy}{dt} = 0.4y(50 - y)$, $L = 50$.

7.11 BC Practice: Euler's Method and Logistic Models

Bài tập luyện tập

Let $y = f(x)$ solve $\dfrac{dy}{dx} = x + 2y$ with $f(0) = 1$. Use Euler's Method with two steps of size 0.25 to approximate $f(0.5)$. Is the approximation an overestimate or an underestimate? Justify.

Lời giải chi tiết

Step 1: slope $0+2(1) = 2$; $y_1 = 1+2(0.25) = 1.5$ at $x = 0.25$. Step 2: slope $0.25+2(1.5) = 3.25$; $y_2 = 1.5 + 3.25(0.25) = 2.3125$ at $x = 0.5$. So $f(0.5) \approx 2.3125$.
$\dfrac{d^2y}{dx^2} = 1 + 2\dfrac{dy}{dx} = 1 + 2(x + 2y) = 1 + 2x + 4y$. Along the path $x \geq 0$ and y starts at 1 and increases (since $\frac{dy}{dx} = x + 2y > 0$ when $x, y > 0$), so $\frac{d^2y}{dx^2} > 0$ on $[0, 0.5]$: the curve is concave up and the estimate is an **underestimate**. (Exact value $\frac{5}{4}e - \frac{1}{2} \approx 2.898$.)

Bài tập luyện tập

Let $y = f(x)$ solve $\dfrac{dy}{dx} = y - x$ with $f(0) = 2$. The tangent line to the graph of f at $x = 0$ is used to approximate $f(0.1)$. Which statement is true?

(A) $f(0.1) \approx 2.2$, an overestimate
(B) $f(0.1) \approx 2.1$, an underestimate
(C) $f(0.1) \approx 2.2$, an underestimate
(D) $f(0.1) \approx 1.8$, an overestimate

Lời giải chi tiết

Slope at $(0, 2)$: $2 - 0 = 2$. Tangent line $y = 2 + 2x$, so $f(0.1) \approx 2.2$. $\dfrac{d^2y}{dx^2} = \dfrac{dy}{dx} - 1 = y - x - 1$, which equals $1 > 0$ at $(0, 2)$: concave up, so the tangent line lies below the curve — an underestimate. **(C)**. (Exact: $f(x) = x + 1 + e^x$, $f(0.1) \approx 2.205$.)

Bài tập luyện tập

Let $y = f(x)$ solve $\dfrac{dy}{dx} = x^2 + y$ with $f(2) = 1$. Use Euler's Method with two steps of size $\Delta x = -0.1$ to approximate $f(1.8)$.

Lời giải chi tiết

Step 1: slope at $(2, 1)$ is $4 + 1 = 5$; $y_1 = 1 + 5(-0.1) = 0.5$ at $x = 1.9$. Step 2: slope at $(1.9, 0.5)$ is $3.61 + 0.5 = 4.11$; $y_2 = 0.5 + 4.11(-0.1) = 0.089$ at $x = 1.8$. So $f(1.8) \approx 0.089$. (Here $\frac{d^2y}{dx^2} = 2x + x^2 + y$, which is $9 > 0$ at $(2, 1)$; the exact value ≈ 0.166 is indeed larger.)

Bài tập luyện tập

A population satisfies $\frac{dP}{dt} = 0.02P - 0.00004P^2$. What is the carrying capacity?

(A) 250
(C) 50
(B) 500
(D) 2000

Lời giải chi tiết

$0.02P - 0.00004P^2 = 0.02P\left(1 - \frac{0.00004}{0.02}P\right) = 0.02P\left(1 - \frac{P}{500}\right)$. $L = 500$. **(B)**. Choice (A) is $L/2$, where growth is fastest.

Bài tập luyện tập

For the population in the previous problem, suppose $P(0) = 700$. Which describes $P(t)$ for $t > 0$?

(A) increasing, with $\lim_{t\to\infty} P(t) = \infty$
(C) increasing, with $\lim_{t\to\infty} P(t) = 700$
(B) decreasing, with $\lim_{t\to\infty} P(t) = 0$
(D) decreasing, with $\lim_{t\to\infty} P(t) = 500$

Lời giải chi tiết

At $P = 700 > 500$, $\frac{dP}{dt} = 0.02(700)(1 - \frac{700}{500}) = 14(-0.4) = -5.6 < 0$, so P decreases. Solutions starting above L decrease toward L without crossing it, so $\lim_{t\to\infty} P(t) = 500$. **(D)**.

Bài tập luyện tập

Solve $\frac{dy}{dt} = 0.5y\left(1 - \frac{y}{40}\right)$ with $y(0) = 10$. Find $y(4)$ and the time at which $y = 20$.

Lời giải chi tiết

Here $k = 0.5$, $L = 40$, $A = \frac{40 - 10}{10} = 3$, so $y(t) = \frac{40}{1 + 3e^{-0.5t}}$. (Derivation: $\int \frac{40\,dy}{y(40-y)} = \int 0.5\,dt$, with $\frac{40}{y(40-y)} = \frac{1}{y} + \frac{1}{40-y}$, gives $\frac{y}{40-y} = Be^{0.5t}$, $B = \frac{1}{3}$.)

$y(4) = \frac{40}{1 + 3e^{-2}} \approx 28.449$. Setting $y = 20$: $1 + 3e^{-0.5t} = 2 \Rightarrow e^{-0.5t} = \frac{1}{3} \Rightarrow t = 2\ln 3 \approx 2.197$.

Bài tập luyện tập

Let $y(t)$ satisfy $\frac{dy}{dt} = \frac{y(10 - y)}{20}$ with $y(0) = 2$.
(a) Find $\lim_{t\to\infty} y(t)$.
(b) Find $\frac{d^2y}{dt^2}$ in terms of y. For what values of y is the graph of y concave up?
(c) Use Euler's Method with two steps of size $\Delta t = 1$ to approximate $y(2)$.
(d) Given that $y(t) < 5$ for $0 \le t \le 2$, is the estimate in (c) an over- or underestimate? Explain.

Lời giải chi tiết

(a) Logistic with $L = 10$; $y(0) = 2 > 0$, so the limit is 10.

(b) $\dfrac{dy}{dt} = \dfrac{10y - y^2}{20}$, so $\dfrac{d^2y}{dt^2} = \dfrac{10 - 2y}{20} \cdot \dfrac{dy}{dt} = \dfrac{(10 - 2y)\, y\, (10 - y)}{400}$. For $0 < y < 10$, $y(10-y) > 0$, so the sign is that of $10 - 2y$: concave up for $0 < y < 5$ (and also for $y > 10$, where both $10 - 2y$ and $10 - y$ are negative).

(c) Step 1: slope $\frac{2 \cdot 8}{20} = 0.8$, $y_1 = 2.8$. Step 2: slope $\frac{2.8 \cdot 7.2}{20} = 1.008$, $y_2 = 2.8 + 1.008 = 3.808$. So $y(2) \approx 3.808$.

(d) On $[0, 2]$, $2 \le y < 5$, so $\frac{d^2y}{dt^2} > 0$: the solution is concave up along the whole path and the Euler estimate is an **underestimate**. (Exact: $y = \frac{10}{1 + 4e^{-t/2}}$, $y(2) \approx 4.046$.)

Bài tập luyện tập

Which differential equation has solutions that, for $y(0) = 1$, approach 12 as $t \to \infty$?

(A) $\dfrac{dy}{dt} = 0.3y(12 - y)$

(B) $\dfrac{dy}{dt} = 12y(0.3 - y)$

(C) $\dfrac{dy}{dt} = 0.3y + 12$

(D) $\dfrac{dy}{dt} = 0.3y(y - 12)$

Lời giải chi tiết

(A) is logistic with $L = 12$; since $0 < 1 < 12$, $y \to 12$. ✓ (B) is logistic with $L = 0.3$, so y would decrease to 0.3. (C) grows without bound. (D) has $\frac{dy}{dt} < 0$ for $0 < y < 12$, so y decreases toward 0. **(A)**.

Bài tập luyện tập

A rumor spreads through a school of 1200 students at a rate proportional to the product of the number y who have heard it and the number who have not. Initially 100 students have heard it, and at that moment the rumor is spreading at 22 students per hour.
(a) Write the differential equation and find the constant of proportionality.
(b) How many students have heard the rumor when it is spreading fastest, and how fast is it spreading then?

Lời giải chi tiết

(a) $\dfrac{dy}{dt} = c\, y\, (1200 - y)$. At $y = 100$: $22 = c(100)(1100) \Rightarrow c = \dfrac{22}{110000} = 0.0002$.

(b) Fastest at $y = \dfrac{1200}{2} = 600$ students, where $\dfrac{dy}{dt} = 0.0002(600)(600) = 72$ students per hour.

7.12 Problem Bank

Bài tập luyện tập

What is the order of the differential equation $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - y = \sin x$?

(A) 1

(B) 2

(C) 3

(D) Cannot be determined

Lời giải chi tiết

The order of a differential equation is the order of the highest derivative present. Here the highest derivative is $\frac{d^2y}{dx^2}$, a second derivative, so the equation is order 2. **(B)**.

Bài tập luyện tập

The rate at which a rumor spreads through a school of N students is proportional to the product of the number of students who have heard it, y, and the number who have not, $(N-y)$. Write a differential equation for $\frac{dy}{dt}$.

Lời giải chi tiết

"Proportional to the product of y and $(N-y)$" translates directly to $\frac{dy}{dt} = ky(N-y)$, where k is the constant of proportionality. This is a logistic-type equation.

Bài tập luyện tập

A tank drains so that the rate of change of the volume of water V is proportional to the square root of V. Which differential equation models this?

(A) $\frac{dV}{dt} = kV$

(B) $\frac{dV}{dt} = k\sqrt{V}$

(C) $\frac{dV}{dt} = kV^2$

(D) $\frac{dV}{dt} = \frac{k}{V}$

Lời giải chi tiết

"Proportional to the square root of V" means the right-hand side is $k\sqrt{V}$, so $\frac{dV}{dt} = k\sqrt{V}$. **(B)**.

Bài tập luyện tập

Show that $y = 5e^{3x} - 4$ is *not* a solution of $\frac{dy}{dx} = 3y$.

Lời giải chi tiết

Differentiate: $\frac{dy}{dx} = 15e^{3x}$. Substitute into the right-hand side: $3y = 3(5e^{3x} - 4) = 15e^{3x} - 12$. Since $15e^{3x} \neq 15e^{3x} - 12$ (the two sides differ by 12 for every x), $y = 5e^{3x} - 4$ is not a solution.

Bài tập luyện tập

Which of the following functions is a solution of $\frac{dy}{dx} = 2xy$?

(A) $y = e^{x^2}$

(B) $y = e^{2x}$

(C) $y = x^2$

(D) $y = 2x^2 + C$

Lời giải chi tiết

Test (A): $y = e^{x^2} \Rightarrow \dfrac{dy}{dx} = 2xe^{x^2} = 2xy$. ✓ This matches exactly. (Quick checks confirm the others fail: for $y = e^{2x}$, $\dfrac{dy}{dx} = 2e^{2x} \neq 2xe^{2x}$ in general; for $y = x^2$, $\dfrac{dy}{dx} = 2x \neq 2x^3$; for $y = 2x^2 + C$, $\dfrac{dy}{dx} = 4x \neq 2x(2x^2 + C)$.) **(A)**.

Bài tập luyện tập

For the differential equation $\dfrac{dy}{dx} = y - x$, compute the slope at each of the points $(0, 0)$, $(0, 1)$, $(0, 2)$, $(1, 0)$, $(1, 1)$, $(2, 0)$, and present the results in a table.

Lời giải chi tiết

Substitute each (x, y) into $y - x$:

(x, y)	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	(x, y)	$(2,0)$
slope	0	1	2	-1	0	slope	-2

Bài tập luyện tập

In the slope field for $\dfrac{dy}{dx} = x - y$, the line along which every segment is horizontal (slope 0) is:

(A) $y = x$

(B) $y = -x$

(C) $y = 0$

(D) $x = 0$

Lời giải chi tiết

Slope is 0 exactly where $x - y = 0$, i.e., $y = x$. **(A)**.

Bài tập luyện tập

A slope field has horizontal segments along the entire x-axis, with segments becoming steeper and positive as y increases above 0, and steeper and negative as y decreases below 0; at any fixed height y, all segments look identical regardless of x. Which differential equation produces this field: (i) $\dfrac{dy}{dx} = x$, (ii) $\dfrac{dy}{dx} = y$, (iii) $\dfrac{dy}{dx} = x + y$, (iv) $\dfrac{dy}{dx} = x - y$?

Lời giải chi tiết

The field is horizontal exactly on $y = 0$ and depends only on y, not x (segments at a given height look the same for every x-value) — this matches $\dfrac{dy}{dx} = y$, equation (ii). Equations (i), (iii), (iv) all involve x explicitly, so their segments would change as x varies at a fixed y, which contradicts the description.

Bài tập luyện tập

For the differential equation $\frac{dy}{dx} = x - y$: **(a)** find the equation of the nullcline (where the slope is 0); **(b)** for a solution curve starting at a point where $y > x$, state whether the curve is initially increasing or decreasing, and describe what happens to the curve as it approaches the nullcline.

Lời giải chi tiết

(a) Set $x - y = 0$, giving the nullcline $y = x$. **(b)** If $y > x$, then $x - y < 0$, so $\frac{dy}{dx} < 0$ and the curve is initially **decreasing**. As the curve approaches the nullcline, $x - y \to 0$, so the slope approaches 0 and the curve flattens out (its rate of decrease slows down as it nears the line $y = x$).

Bài tập luyện tập

Use Euler's Method with step size $\Delta x = 0.5$ to approximate $y(2)$ for $\frac{dy}{dx} = x - y$, given $y(0) = 2$. Show your work in a table with four steps.

Lời giải chi tiết

With $f(x, y) = x - y$:

n	x_n	y_n	$f(x_n, y_n) = x_n - y_n$	$y_{n+1} = y_n + f(0.5)$
0	0	2	-2	$2 + (-2)(0.5) = 1$
1	0.5	1	-0.5	$1 + (-0.5)(0.5) = 0.75$
2	1.0	0.75	0.25	$0.75 + 0.25(0.5) = 0.875$
3	1.5	0.875	0.625	$0.875 + 0.625(0.5) = 1.1875$

So $y(2) \approx 1.1875$.

Bài tập luyện tập

The exact solution to a differential equation is known to be concave down throughout the interval used for an Euler's Method approximation, with x increasing at each step. Compared to the exact solution, the Euler approximation will:

(A) Always overestimate

(B) Always underestimate

(C) Match exactly

(D) Cannot be determined

Lời giải chi tiết

If the exact solution is concave down, its tangent lines lie above the curve. Since Euler's Method steps along tangent-line directions, each step lands above the true curve, so the approximation always overestimates. **(A)**.

Bài tập luyện tập

Given $\frac{dy}{dx} = 2y$, $y(0) = 1$, use Euler's Method with $\Delta x = 0.1$ to estimate $y(0.2)$. Show two steps.

Lời giải chi tiết

$f(x, y) = 2y$. Step 0: $f(0, 1) = 2$, so $y_1 = 1 + 2(0.1) = 1.2$ at $x_1 = 0.1$. Step 1: $f(0.1, 1.2) = 2.4$, so $y_2 = 1.2 + 2.4(0.1) = 1.44$ at $x_2 = 0.2$. Euler estimate: $y(0.2) \approx 1.44$. (The exact solution is $y = e^{2x}$, giving $y(0.2) = e^{0.4} \approx 1.4918$, showing the approximation is reasonably close but slightly low.)

Bài tập luyện tập

The general solution of $\frac{dy}{dx} = xy$ is:

(A) $y = Ce^{x^2/2}$

(B) $y = Ce^{x^2}$

(C) $y = \frac{Cx^2}{2}$

(D) $y = Ce^{2x}$

Lời giải chi tiết

Separate: $\frac{dy}{y} = x\,dx \Rightarrow \ln|y| = \frac{x^2}{2} + C_1 \Rightarrow y = Ce^{x^2/2}$. **(A)**.

Bài tập luyện tập

Solve the initial value problem $\frac{dy}{dx} = \frac{6x^2}{y}$, $y(0) = -4$. Express y explicitly in terms of x.

Lời giải chi tiết

Separate: $y\,dy = 6x^2\,dx \Rightarrow \frac{y^2}{2} = 2x^3 + C \Rightarrow y^2 = 4x^3 + K$. Using $y(0) = -4$: $16 = 0 + K \Rightarrow K = 16$, so $y^2 = 4x^3 + 16$. Since $y(0) = -4 < 0$, take the negative branch:

$$y = -\sqrt{4x^3 + 16}.$$

Check: $\frac{dy}{dx} = -\frac{12x^2}{2\sqrt{4x^3+16}} = -\frac{6x^2}{\sqrt{4x^3+16}} = \frac{6x^2}{-\sqrt{4x^3+16}} = \frac{6x^2}{y}$. ✓

Bài tập luyện tập

Solve the initial value problem $\frac{dy}{dx} = \frac{y}{x+1}$, $y(0) = 3$. Express y explicitly in terms of x.

Lời giải chi tiết

Separate: $\dfrac{dy}{y} = \dfrac{dx}{x+1} \Rightarrow \ln|y| = \ln|x+1| + C \Rightarrow y = A(x+1)$. Using $y(0) = 3$: $3 = A(1) \Rightarrow A = 3$, so

$$y = 3(x+1) = 3x + 3.$$

Check: $\dfrac{dy}{dx} = 3$, and $\dfrac{y}{x+1} = \dfrac{3x+3}{x+1} = 3$. ✓

Bài tập luyện tập

Solve the initial value problem $\dfrac{dy}{dx} = \dfrac{x^2}{3y^2 - 2y}$, $y(0) = 1$, leaving the answer in implicit form.

Lời giải chi tiết

Separate: $(3y^2 - 2y)\,dy = x^2\,dx \Rightarrow \int (3y^2 - 2y)\,dy = \int x^2\,dx \Rightarrow y^3 - y^2 = \dfrac{x^3}{3} + C$. Using $x = 0$, $y = 1$: $1 - 1 = 0 = 0 + C \Rightarrow C = 0$. Implicit solution:

$$y^3 - y^2 = \frac{x^3}{3}, \qquad \text{or} \qquad 3y^3 - 3y^2 - x^3 = 0.$$

Check by implicit differentiation: $9y^2y' - 6yy' = 3x^2 \Rightarrow y'(9y^2 - 6y) = 3x^2 \Rightarrow y' = \dfrac{3x^2}{9y^2 - 6y} = \dfrac{x^2}{3y^2 - 2y}$, matching the original equation. ✓

Bài tập luyện tập

After separating variables in a differential equation, a student needs to evaluate $\displaystyle\int \frac{dy}{y}$. This integral equals:

(A) $\ln|y| + C$

(B) $\dfrac{y^2}{2} + C$

(C) $\dfrac{1}{y} + C$

(D) $e^y + C$

Lời giải chi tiết

$\displaystyle\int \frac{dy}{y} = \ln|y| + C$, by the standard antiderivative of $1/y$ (absolute value required since y may be negative). **(A)**.

Bài tập luyện tập

A student solves $\dfrac{dy}{dx} = -\dfrac{x}{y}$ and writes the general solution as $y = x^2 + C$. Identify the student's error and give the correct general solution.

Lời giải chi tiết

The student appears to have ignored the $\frac{1}{y}$ factor and simply integrated $dy = -x\,dx$ as if the equation were $\frac{dy}{dx} = -x$, skipping the separation step entirely. The correct approach: separate as $y\,dy = -x\,dx$, then integrate: $\frac{y^2}{2} = -\frac{x^2}{2} + C \Rightarrow x^2 + y^2 = K$ (where $K = 2C$) — a family of circles centered at the origin, not a family of parabolas.

Bài tập luyện tập

A colony of bacteria grows according to $\frac{dP}{dt} = kP$. Initially there are 200 bacteria, and after 4 hours there are 650. **(a)** Find k. **(b)** Find the population after 10 hours.

Lời giải chi tiết

(a) $P(t) = 200e^{kt}$. Using $P(4) = 650$: $e^{4k} = 650/200 = 3.25 \Rightarrow k = \frac{\ln 3.25}{4} \approx \frac{1.17865}{4} \approx 0.2947$ per hour.

(b) $P(10) = 200e^{10k} = 200\,(3.25)^{10/4} = 200\,(3.25)^{2.5} \approx 200(19.04) \approx 3808$ bacteria.

Bài tập luyện tập

If a radioactive substance decays according to $\frac{dy}{dt} = ky$ with $k = -0.1$ per year, its half-life is closest to:

(A) 5 years
(B) 6.93 years
(C) 10 years
(D) 14 years

Lời giải chi tiết

$t_{1/2} = \frac{\ln 2}{|k|} = \frac{0.6931}{0.1} \approx 6.93$ years. **(B).**

Bài tập luyện tập

A radioactive isotope has a half-life of 8 years. Starting with 50 grams, how much remains after 20 years?

Lời giải chi tiết

Number of half-lives elapsed: $20/8 = 2.5$. Remaining amount:

$$50\left(\frac{1}{2}\right)^{2.5} = \frac{50}{2^{2.5}} = \frac{50}{4\sqrt{2}} = \frac{12.5}{\sqrt{2}} \approx 8.84 \text{ grams.}$$

Bài tập luyện tập

A metal rod at $150°$C is placed in a room kept at $25°$C. After 5 minutes, the rod's temperature is $100°$C. **(a)** Find k in $\frac{dT}{dt} = k(T - 25)$. **(b)** Find the rod's temperature after 15 minutes.

Lời giải chi tiết

(a) $T(t) = 25 + Ae^{kt}$, $A = 150 - 25 = 125$. Using $T(5) = 100$: $75 = 125e^{5k} \Rightarrow e^{5k} = \frac{3}{5} \Rightarrow k = \frac{\ln(3/5)}{5} \approx -0.1022$ per minute.

(b) Since $15 = 3 \times 5$, $e^{15k} = (e^{5k})^3 = \left(\frac{3}{5}\right)^3 = \frac{27}{125} = 0.216$.

$$T(15) = 25 + 125(0.216) = 25 + 27 = 52°\text{C}.$$

Bài tập luyện tập

In the model $\frac{dT}{dt} = k(T - T_{amb})$, for an object that is warmer than its surroundings and cooling over time, the constant k must be:

(A) Positive
(B) Negative
(C) Zero
(D) Cannot be determined

Lời giải chi tiết

Since $T > T_{amb}$, we have $T - T_{amb} > 0$; for the object to cool, $\frac{dT}{dt}$ must be negative, which requires $k < 0$. **(B)**.

Bài tập luyện tập

For the logistic differential equation $\frac{dP}{dt} = 0.4P\left(1 - \frac{P}{2000}\right)$, verify that $P = 0$ and $P = 2000$ are equilibrium solutions, and find the population at which the growth rate is maximized, along with that maximum rate.

Lời giải chi tiết

At $P = 0$: $0.4(0)(1 - 0) = 0$. ✓ At $P = 2000$: $0.4(2000)(1 - 1) = 0$. ✓ Both are equilibria. Maximum growth occurs at $P = L/2 = 1000$; the maximum rate is $\frac{kL}{4} = \frac{0.4(2000)}{4} = 200$. Check directly: at $P = 1000$, $0.4(1000)(1 - 0.5) = 0.4(1000)(0.5) = 200$. ✓

Bài tập luyện tập

For the logistic model $\frac{dP}{dt} = 0.05P\left(1 - \frac{P}{8000}\right)$, the carrying capacity of this population is:

(A) 0.05
(B) 8000
(C) 4000
(D) 400

Lời giải chi tiết

The carrying capacity L is the value that makes the $(1 - P/L)$ factor vanish; here $L = 8000$. **(B)**.

Bài tập luyện tập

A population modeled by $\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$ currently has $P < L/2$. Is the growth *rate* $\frac{dP}{dt}$ currently increasing or decreasing as P increases toward $L/2$? Explain using the shape of $\frac{dP}{dt}$ as a function of P.

Lời giải chi tiết

As a function of P, $\frac{dP}{dt} = kP - \frac{k}{L}P^2$ is a downward-opening parabola in P with vertex (maximum) at $P = L/2$. For $P < L/2$, the parabola is still rising, so $\frac{dP}{dt}$ is **increasing** as P increases toward $L/2$ — the population's growth rate keeps speeding up until P reaches $L/2$, after which it begins to slow down.

Bài tập luyện tập

Show that the equation $x^2 + y^2 = 25$ implicitly defines a solution to $\frac{dy}{dx} = -\frac{x}{y}$ (for $y \neq 0$), by differentiating implicitly.

Lời giải chi tiết

Differentiate both sides of $x^2 + y^2 = 25$ with respect to x: $2x + 2yy' = 0 \Rightarrow y' = -\frac{2x}{2y} = -\frac{x}{y}$, which is exactly the given differential equation. ✓

Bài tập luyện tập

Which of the following differential equations is *not* separable?

(A) $\frac{dy}{dx} = xy$

(B) $\frac{dy}{dx} = x + y$

(C) $\frac{dy}{dx} = \frac{x}{y}$

(D) $\frac{dy}{dx} = e^x y^2$

Lời giải chi tiết

A separable equation must factor as $g(x)h(y)$. Choices (A), (C), (D) all factor this way (xy, $x \cdot \frac{1}{y}$, $e^x \cdot y^2$), but $x+y$ is a *sum*, not a product, and cannot be split into a function of x times a function of y. **(B)**.

Bài tập luyện tập

Solve the initial value problem $\frac{dy}{dx} = y\cos x$, $y(0) = 1$. Express y explicitly in terms of x.

Lời giải chi tiết

Separate: $\dfrac{dy}{y} = \cos x\,dx \Rightarrow \ln|y| = \sin x + C \Rightarrow y = Ae^{\sin x}$. Using $y(0) = 1$: $1 = Ae^0 = A \Rightarrow A = 1$, so

$$y = e^{\sin x}.$$

Check: $\dfrac{dy}{dx} = \cos x \cdot e^{\sin x} = y\cos x$. ✓

Bài tập luyện tập

For the differential equation $\dfrac{dy}{dx} = \dfrac{x}{4y}$, find the general solution and describe the family of curves it represents.

Lời giải chi tiết

Separate: $4y\,dy = x\,dx \Rightarrow 2y^2 = \dfrac{x^2}{2} + C \Rightarrow 4y^2 - x^2 = K$ (where $K = 4C$). This is a family of hyperbolas centered at the origin, opening along the y-axis (for $K > 0$) or the x-axis (for $K < 0$).

Applications of Integration

Mục tiêu Unit: Giá trị trung bình của hàm số và Định lý Giá trị Trung bình cho tích phân; chuyển động thẳng (vị trí – vận tốc – gia tốc) bằng tích phân, phân biệt độ dời (displacement) và quãng đường đi được (total distance); Định lý Biến thiên Ròng (Net Change Theorem) và các bài toán tích lũy tốc độ vào/ra (bồn nước, dữ liệu dạng bảng, hàm từng khúc); diện tích giữa hai đường cong theo biến x lẫn biến y, kể cả khi hai đường cắt nhau nhiều lần; thể tích khối có thiết diện đã biết (hình vuông, hình chữ nhật, tam giác đều, nửa hình tròn); và phương pháp đĩa (disk) cùng phương pháp vành khăn (washer) khi quay quanh trục hoành, trục tung, và các đường thẳng khác.

8.1 Average Value of a Function and the Mean Value Theorem for Integrals

8.1.1 Defining the average value of a function

For finitely many numbers $y_1, y_2, \dots, y_n$, the average is $\frac{y_1+y_2+\cdots+y_n}{n}$. Calculus extends this idea to a function f that is continuous on a closed interval $[a, b]$, where there are infinitely many values $f(x)$ to average over. Partition $[a, b]$ into n equal subintervals of width $\Delta x = \frac{b-a}{n}$ and sample f at a point x_i^* in each subinterval; the average of these n sampled values is

$$\frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n} = \frac{1}{b-a} \sum_{i=1}^{n} f(x_i^*)\, \Delta x,$$

since $n = \frac{b-a}{\Delta x}$. As $n \to \infty$ the Riemann sum $\sum f(x_i^*)\Delta x$ converges to $\int_a^b f(x)\, dx$, which motivates the definition:

Average Value of a Function

If f is continuous (or at least integrable) on $[a, b]$, the **average value of f on $[a, b]$** is

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

Equivalently, $\displaystyle\int_a^b f(x)\, dx = f_{\text{avg}} \cdot (b-a)$.

Geometric meaning. If $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\, dx$ is the area of the region between the graph of f and the x-axis. The equation $\int_a^b f(x)\, dx = f_{\text{avg}}(b-a)$ says that f_{avg} is exactly the **height of the rectangle with base $[a, b]$ that has the same area** as that region — a single flat height that "levels out" all the peaks and valleys of f while preserving the total accumulated area.

VN **GIẢI THÍCH TIẾNG VIỆT**

Giá trị trung bình của hàm số $f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x)\, dx$ là phép mở rộng của khái niệm "trung bình cộng" sang trường hợp có vô số giá trị (một cho mỗi điểm x trong $[a, b]$), thay vì chỉ hữu hạn số hạng. Về mặt hình học, đây chính là **chiều cao của hình chữ nhật** có đáy $[a, b]$ sao cho

diện tích hình chữ nhật đó bằng đúng diện tích miền giữa đồ thị f và trục hoành trên $[a, b]$ — nói cách khác, ta "san phẳng" các đoạn lên xuống của đồ thị thành một mức chiều cao không đổi mà vẫn giữ nguyên tổng diện tích.

8.1.2 The Mean Value Theorem for Integrals

Because f_{avg} is defined as an average of the continuous function's own values, it is natural to ask whether f actually *attains* this average value somewhere on $[a, b]$. The answer is yes, guaranteed by continuity.

Mean Value Theorem for Integrals

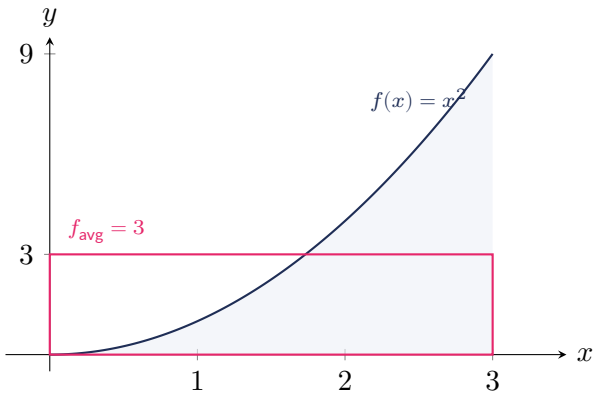
If f is continuous on $[a, b]$, then there exists at least one number c in $[a, b]$ such that

$$f(c) = f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx, \qquad \text{equivalently} \qquad \int_a^b f(x)\,dx = f(c)\,(b-a).$$

Sketch of proof: by the Extreme Value Theorem, f attains a minimum m and maximum M on $[a, b]$, so $m \le f(x) \le M$ for all $x \in [a, b]$. Integrating this inequality over $[a, b]$ gives $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$, i.e. $m \le f_{\text{avg}} \le M$. Since f is continuous, the Intermediate Value Theorem guarantees some $c \in [a, b]$ with $f(c) = f_{\text{avg}}$.

VN **GIẢI THÍCH TIẾNG VIỆT**

Định lý Giá trị Trung bình cho tích phân (MVT for Integrals) khẳng định rằng nếu f liên tục trên $[a, b]$ thì tồn tại ít nhất một điểm c sao cho $f(c)$ *bằng đúng* giá trị trung bình f_{avg} — nghĩa là hàm số thực sự "đạt được" giá trị trung bình của chính nó tại một thời điểm nào đó, chứ không chỉ là một con số trừu tượng. Ý tưởng chứng minh: vì f liên tục nên bị chặn bởi giá trị nhỏ nhất m và lớn nhất M trên $[a, b]$; lấy tích phân bất đẳng thức $m \le f(x) \le M$ suy ra f_{avg} cũng nằm giữa m và M; theo Định lý Giá trị Trung gian (IVT), f phải đạt giá trị f_{avg} tại một điểm c nào đó.



The shaded region under $f(x) = x^2$ on $[0, 3]$ and the pink rectangle of height $f_{\text{avg}} = 3$ and base 3 have exactly the same area, 9.

Ví dụ mẫu · Average value and the guaranteed value of c

Let $f(x) = x^2$ on $[0, 3]$. Find the average value of f on this interval, and find the value(s) of c guaranteed by the Mean Value Theorem for Integrals.

Average value:

$$f_{\text{avg}} = \frac{1}{3-0}\int_0^3 x^2\,dx = \frac{1}{3}\left[\frac{x^3}{3}\right]_0^3 = \frac{1}{3}\cdot\frac{27}{3} = \frac{1}{3}\cdot 9 = 3.$$

Finding c: we need $f(c) = 3$, i.e. $c^2 = 3$, so $c = \pm\sqrt{3}$. Only $c = \sqrt{3} \approx 1.732$ lies in $[0, 3]$ (the negative root is outside the interval), so the theorem is satisfied by $\boxed{c = \sqrt{3}}$.

(!) Lỗi thường gặp: Đừng nhầm **giá trị trung bình** f_{avg} với **trung bình cộng của hai giá trị đầu mút** $\frac{f(a)+f(b)}{2}$ — hai đại lượng này chỉ trùng nhau khi f là hàm bậc nhất (tuyến tính). Với $f(x) = x^2$ trên $[0, 3]$: $\frac{f(0)+f(3)}{2} = \frac{0+9}{2} = 4.5$, hoàn toàn khác với $f_{\text{avg}} = 3$ tính đúng theo công thức tích phân. Một lỗi khác thường gặp là **quên chia cho** $(b-a)$ — nhớ rằng $\int_a^b f(x)\,dx$ chỉ là diện tích (hoặc diện tích có dấu), còn giá trị trung bình luôn cần chia thêm cho độ dài khoảng $b-a$.

8.2 Motion Along a Line

8.2.1 Position, velocity, and acceleration as integrals

If a particle moves along a line with acceleration $a(t)$ and velocity $v(t)$, both continuous, the Net Change Theorem (Section 3) lets us reconstruct velocity from acceleration and position from velocity:

Reconstructing motion by integration

$$v(t) = v(t_0) + \int_{t_0}^{t} a(\tau)\,d\tau, \qquad s(t) = s(t_0) + \int_{t_0}^{t} v(\tau)\,d\tau.$$

That is: velocity at time t equals the initial velocity plus the net accumulated change in velocity (the integral of acceleration); position at time t equals the initial position plus the net accumulated change in position (the integral of velocity).

Two very different quantities can be extracted from $v(t)$ over an interval $[a, b]$:

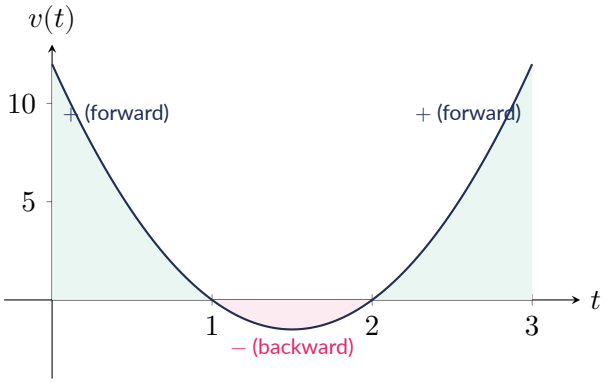
- **Net displacement** (signed change in position) $= \int_a^b v(t)\,dt = s(b) - s(a)$. This allows positive and negative contributions (forward and backward motion) to cancel.
- **Total distance traveled** $= \int_a^b |v(t)|\,dt$. This treats every interval of motion as a positive contribution regardless of direction, so backward motion *adds* to the total rather than canceling forward motion.

When $v(t)$ does not change sign on $[a, b]$, the two quantities are equal (up to sign). When $v(t)$ changes sign, they differ, and computing total distance requires splitting the interval at every zero of $v(t)$, integrating v (not $|v|$) on each piece, and adding the *absolute values* of those pieces.

VN **GIẢI THÍCH TIẾNG VIỆT**

Vận tốc $v(t)$ được dựng lại từ **gia tốc** $a(t)$ bằng tích phân, và **vị trí** $s(t)$ được dựng lại từ vận tốc theo đúng cách tương tự — đây chính là ứng dụng trực tiếp của Định lý Biến thiên Ròng. Điểm mấu chốt cần phân biệt: **độ dời (displacement)** $= \int_a^b v(t)\,dt$ là sự thay đổi vị trí có dấu (có thể dương, âm, hoặc bằng 0 nếu vật quay về điểm xuất phát), trong khi **quãng đường đi được**

(total distance) $= \int_a^b |v(t)|\,dt$ luôn không âm và luôn *cộng dồn* mọi đoạn chuyển động, bất kể chiều. Khi vận tốc đổi dấu trên đoạn đang xét, hai đại lượng này chắc chắn khác nhau.



$v(t) = 6t^2 - 18t + 12$ on $[0, 3]$: mint-shaded regions (where $v > 0$) and the pink-shaded dip (where $v < 0$) all contribute positively to total distance, but the pink region subtracts from net displacement.

Ví dụ mẫu · Displacement vs. total distance

A particle moves along a line with velocity $v(t) = 6t^2 - 18t + 12$ (meters per second) for $0 \le t \le 3$. Find (a) the particle's net displacement, and (b) the total distance it travels, on $[0, 3]$.

Step 1 — locate direction changes. Factor: $v(t) = 6(t^2 - 3t + 2) = 6(t-1)(t-2)$, so $v(t) = 0$ at $t = 1$ and $t = 2$. Testing signs: $v(0) = 12 > 0$, $v(1.5) = 6(0.25)(-0.5) = -0.75 < 0$, $v(2.5) = 6(1.5)(0.5) = 4.5 > 0$. So $v > 0$ on $[0, 1)$, $v < 0$ on $(1, 2)$, $v > 0$ on $(2, 3]$.

Step 2 — an antiderivative. $\int v(t)\,dt = 2t^3 - 9t^2 + 12t = P(t)$. Evaluate: $P(0) = 0$, $P(1) = 2 - 9 + 12 = 5$, $P(2) = 16 - 36 + 24 = 4$, $P(3) = 54 - 81 + 36 = 9$.

(a) Net displacement $= \int_0^3 v(t)\,dt = P(3) - P(0) = 9 - 0 = \boxed{9 \text{ meters}}$.

(b) Total distance: sum the absolute value of the net change on each monotonic-sign piece:

$$|P(1)-P(0)| + |P(2)-P(1)| + |P(3)-P(2)| = |5-0| + |4-5| + |9-4| = 5+1+5 = \boxed{11 \text{ meters}}.$$

Notice $9 \neq 11$: on $(1, 2)$ the particle backtracks 1 meter, which *reduces* net displacement but still *adds* to total distance.

(!) Lỗi thường gặp: Lỗi cực kỳ phổ biến: tính $\int_a^b v(t)\,dt$ rồi coi đó luôn là quãng đường đi được. Điều này **chỉ đúng khi** $v(t)$ **không đổi dấu** trên $[a, b]$. Nếu $v(t)$ đổi dấu, $\int_a^b v(t)\,dt$ chỉ cho **độ dời** (các đoạn tiến và lùi triệt tiêu lẫn nhau); muốn tính **quãng đường**, bắt buộc phải: (1) tìm mọi nghiệm của $v(t) = 0$ trong $[a, b]$; (2) tách tích phân tại các nghiệm đó; (3) lấy **trị tuyệt đối** của từng mảnh rồi cộng lại — tức là tính $\int_a^b |v(t)|\,dt$, không phải $\left|\int_a^b v(t)\,dt\right|$ (hai biểu thức này nói chung khác nhau).

8.3 Net Change and Accumulation in Applied Contexts

8.3.1 The Net Change Theorem

The Fundamental Theorem of Calculus, applied to a rate function f', gives one of the most useful results in applied calculus:

Net Change Theorem

If F has a continuous derivative $F' = f$ on $[a, b]$, then

$$\int_a^b f(x)\,dx = \int_a^b F'(x)\,dx = F(b) - F(a).$$

In words: **the definite integral of a rate of change equals the total (net) change in the quantity itself** over that interval. Whatever F represents — position, volume, population, revenue — integrating its *rate* F' over $[a, b]$ recovers exactly how much F changed, without needing to know F's actual value at either endpoint (only the change).

This single idea unifies displacement-from-velocity (Section 2) with a huge range of applied "accumulation" problems: whenever a quantity $Q(t)$ changes at some given rate $R(t) = Q'(t)$, the net change in Q over $[a, b]$ is $\int_a^b R(t)\,dt$, and the value at a later time is $Q(b) = Q(a) + \int_a^b R(t)\,dt$.

8.3.2 Rate in minus rate out

A common applied setting has a quantity (water in a tank, cars in a parking garage, chemical in a solution) governed by two competing rates: an **inflow rate** $R_{\text{in}}(t)$ and an **outflow rate** $R_{\text{out}}(t)$. The **net rate of change** is $R_{\text{net}}(t) = R_{\text{in}}(t) - R_{\text{out}}(t)$, and the Net Change Theorem gives

$$Q(b) = Q(a) + \int_a^b \left[R_{\text{in}}(t) - R_{\text{out}}(t)\right] dt.$$

Since $Q'(t) = R_{\text{net}}(t)$, the usual first-derivative-test logic applies directly to Q: Q is **increasing** exactly where $R_{\text{net}}(t) > 0$ (inflow exceeds outflow) and **decreasing** where $R_{\text{net}}(t) < 0$. A local maximum of Q occurs where R_{net} changes from positive to negative; a local minimum occurs where R_{net} changes from negative to positive.

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Định lý Biến thiên Ròng (Net Change Theorem) là cách phát biểu lại FTC theo ngôn ngữ ứng dụng: tích phân của *tốc độ thay đổi* trên $[a, b]$ chính là *tổng lượng thay đổi* của đại lượng gốc, bất kể đại lượng đó là gì (thể tích nước, dân số, doanh thu...). Trong bài toán "tốc độ vào trừ tốc độ ra" ($R_{\text{in}} - R_{\text{out}}$), lượng tích lũy $Q(t)$ tăng khi tốc độ ròng dương (vào nhiều hơn ra) và giảm khi tốc độ ròng âm — vì vậy cực trị của $Q(t)$ xảy ra đúng tại thời điểm tốc độ ròng *đổi dấu*, hoàn toàn tương tự cách tìm cực trị bằng đạo hàm cấp một.

Ví dụ mẫu · A water tank with competing inflow and outflow

Water flows into a tank at rate $R_{\text{in}}(t) = 20 - 2t$ liters per minute and is pumped out at a constant rate $R_{\text{out}}(t) = 8$ liters per minute, for $0 \le t \le 10$ minutes. At $t = 0$ the tank holds 50 liters. (a) At what time is the volume of water in the tank at a maximum, and what is that maximum volume? (b) How much water is in the tank at $t = 10$?

Net rate: $R_{\text{net}}(t) = (20 - 2t) - 8 = 12 - 2t$. This is positive (tank filling) for $t < 6$ and negative (tank draining) for $t > 6$, so the volume $Q(t)$ increases then decreases — a maximum occurs at $\boxed{t = 6}$ minutes, where the net rate switches from positive to negative.

(a) Maximum volume:

$$Q(6) = 50 + \int_0^6 (12 - 2t)\,dt = 50 + \Big[12t - t^2\Big]_0^6 = 50 + (72 - 36) = 50 + 36 = \boxed{86 \text{ liters}}.$$

(b) Volume at $t = 10$:

$$Q(10) = 50 + \int_0^{10} (12 - 2t)\,dt = 50 + \Big[12t - t^2\Big]_0^{10} = 50 + (120 - 100) = 50 + 20 = \boxed{70 \text{ liters}}.$$

The tank fills from 50 up to a peak of 86 liters at $t = 6$, then drains back down to 70 liters by $t = 10$ — consistent with $70 < 86$ but still above the starting volume of 50.

8.3.3 Rates given by a table or a piecewise function

On the AP exam, a rate is often given not as a clean formula but as a **table of values** at unevenly spaced times, or as a **piecewise function**. For tabular data, the definite integral (total accumulated change) is estimated using a **trapezoidal sum**: on each subinterval $[t_i, t_{i+1}]$, approximate the area under the rate curve by a trapezoid,

$$\int_{t_i}^{t_{i+1}} R(t)\,dt \approx (t_{i+1} - t_i) \cdot \frac{R(t_i) + R(t_{i+1})}{2},$$

and sum these trapezoids across all subintervals — the subintervals do *not* need to be equally spaced. For a piecewise rate function, split the integral at every breakpoint and integrate each piece with its own formula.

Ví dụ mẫu · Rate given by a table, and rate given piecewise

(a) Table. During a 6-hour storm, the rate at which water flows into a retention pond, $R(t)$ (in thousands of liters per hour), is recorded at selected times:

t (hours)	0	2	5	7	8
$R(t)$ (thousand L/hr)	3	9	15	6	2

Use a trapezoidal sum with these four subintervals to estimate $\displaystyle\int_0^8 R(t)\,dt$, the total inflow (in thousand liters) over the 8 hours.

$$\int_0^8 R(t)\,dt \approx (2-0)\cdot\frac{3+9}{2} + (5-2)\cdot\frac{9+15}{2} + (7-5)\cdot\frac{15+6}{2} + (8-7)\cdot\frac{6+2}{2}$$
$$= 2(6) + 3(12) + 2(10.5) + 1(4) = 12 + 36 + 21 + 4 = \boxed{73 \text{ thousand liters}}.$$

(b) Piecewise. A water tower's net fill rate (in thousand gallons per minute) is

$$R(t) = \begin{cases} 5, & 0 \le t < 3 \\ -2t + 11, & 3 \le t \le 8 \end{cases}$$

(Notice R is continuous at $t = 3$: both pieces give $R(3) = 5$.) Find the total net change in the tower's volume over $[0, 8]$.

$$\int_0^8 R(t)\,dt = \int_0^3 5\,dt + \int_3^8 (-2t + 11)\,dt = \Big[5t\Big]_0^3 + \Big[-t^2 + 11t\Big]_3^8$$
$$= 15 + \big[(-64 + 88) - (-9 + 33)\big] = 15 + (24 - 24) = \boxed{15 \text{ thousand gallons}}.$$

The second piece happens to contribute a net change of exactly 0 (the rate decreases linearly from 5 at $t = 3$ down through 0 at $t = 5.5$ to -5 at $t = 8$, and the positive and negative areas cancel), so the entire net gain comes from the first three minutes.

8.4 Area Between Two Curves

8.4.1 Integrating with respect to x

If $f(x) \geq g(x)$ on $[a, b]$, the vertical distance between the two curves at a given x is $f(x) - g(x)$. Summing these thin vertical strips of width dx over $[a, b]$ gives the area of the region enclosed between the curves:

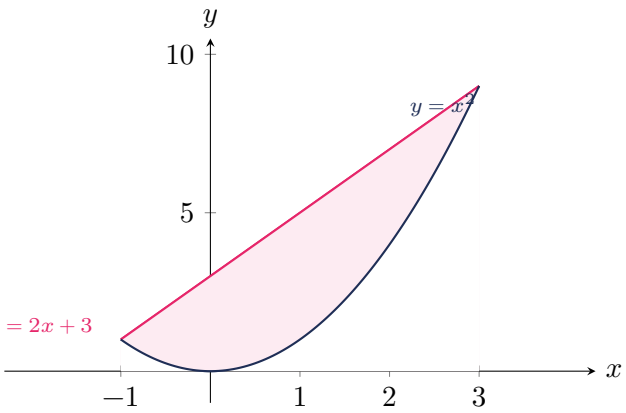
Area Between Curves — with respect to x

$$A = \int_a^b \left[\, \text{top}(x) - \text{bottom}(x) \,\right] dx,$$

where $\text{top}(x)$ and $\text{bottom}(x)$ are whichever of the two functions has the larger, resp. smaller, y-value on (a, b). If the two curves are given without explicit bounds, the bounds a and b are usually their **points of intersection**, found by solving $f(x) = g(x)$.

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Diện tích giữa hai đường cong theo biến x luôn là "đường trên trừ đường dưới, rồi lấy tích phân": $A = \int_a^b [\text{top}(x) - \text{bottom}(x)]\, dx$. Bước quan trọng nhất — và dễ bị bỏ sót nhất — là **tìm giao điểm** của hai đường trước khi thiết lập tích phân, vì các giao điểm chính là cận a, b (trừ khi đề bài đã cho sẵn cận). Sau khi có cận, cần kiểm tra (bằng cách thử một điểm bất kỳ nằm giữa a và b) xem đường nào thực sự nằm trên để đặt đúng thứ tự trừ.



The shaded region between $y = 2x + 3$ (top) and $y = x^2$ (bottom), bounded by their intersection points $x = -1$ and $x = 3$.

Ví dụ mẫu · Area between a parabola and a line

Find the area of the region enclosed by $y = x^2$ and $y = 2x + 3$.

Step 1 — intersections: $x^2 = 2x + 3 \Rightarrow x^2 - 2x - 3 = 0 \Rightarrow (x - 3)(x + 1) = 0 \Rightarrow x = -1,\ 3.$

Step 2 — which curve is on top? Test $x = 0$: $y = 2(0) + 3 = 3$ versus $y = 0^2 = 0$. So the line $y = 2x + 3$ is on top throughout $(-1, 3)$.

Step 3 — integrate:

$$A = \int_{-1}^{3} \left[(2x + 3) - x^2\right] dx = \left[x^2 + 3x - \frac{x^3}{3}\right]_{-1}^{3}.$$

At $x=3$: $9+9-9=9$. At $x=-1$: $1-3-\left(-\frac{1}{3}\right)=1-3+\frac{1}{3}=-\frac{5}{3}$.

$$A=9-\left(-\frac{5}{3}\right)=9+\frac{5}{3}=\frac{27}{3}+\frac{5}{3}=\boxed{\frac{32}{3}}.$$

(!) Lỗi thường gặp: Đừng thiết lập tích phân diện tích trước khi **tìm giao điểm**. Nếu đề không cho sẵn cận a, b, hãy giải $f(x)=g(x)$ trước tiên — đây chính là cận tích phân. Lỗi thứ hai thường gặp: đặt sai thứ tự trừ (lấy "đường dưới trừ đường trên"), khiến kết quả ra **âm**. Nếu tính ra diện tích âm, đó là dấu hiệu rõ ràng cần đảo lại thứ tự hai hàm — luôn thử một giá trị x ở giữa khoảng $[a,b]$ để xác định chắc chắn đường nào thực sự nằm trên trước khi viết tích phân.

8.4.2 Integrating with respect to y

Some regions are far easier to describe as functions of y — bounded left and right by curves $x=p(y)$ and $x=q(y)$ — than as functions of x. This happens, for instance, whenever a curve like $x=y^2$ is not a function of x (it fails the vertical line test) but *is* a function of y. The same idea applies with the roles of the axes swapped:

Area Between Curves — with respect to y

$$A=\int_c^d \left[\,\text{right}(y)-\text{left}(y)\,\right]dy,$$

where right(y) and left(y) are whichever curve has the larger, resp. smaller, x-value, and $[c,d]$ is the y-range of the region (again typically found from the curves' intersection points, but now solved for y).

VN **GIẢI THÍCH TIẾNG VIỆT**

Khi miền cần tính diện tích bị chặn **trái–phải** bởi hai đường cong (thay vì trên–dưới), việc lấy tích phân theo y thường đơn giản hơn nhiều so với cố gắng lấy tích phân theo x (nhất là khi một trong hai đường không phải là hàm số của x, ví dụ $x=y^2$). Công thức hoàn toàn tương tự, chỉ đổi vai trò: $A=\int_c^d[\text{right}(y)-\text{left}(y)]\,dy$, với cận c, d tìm được bằng cách giải giao điểm theo biến y.

Ví dụ mẫu · Area using horizontal strips

Find the area of the region enclosed by $x=y^2$ and $x=y+2$.

Step 1 — intersections (solve for y): $y^2=y+2\Rightarrow y^2-y-2=0\Rightarrow(y-2)(y+1)=0\Rightarrow y=-1,\,2$.

Step 2 — which curve is to the right? Test $y=0$: $x=y+2=2$ versus $x=y^2=0$. So $x=y+2$ is to the right (larger x) throughout $(-1,2)$.

Step 3 — integrate with respect to y:

$$A=\int_{-1}^{2}\left[(y+2)-y^2\right]dy=\left[\frac{y^2}{2}+2y-\frac{y^3}{3}\right]_{-1}^{2}.$$

At $y=2$: $2+4-\frac{8}{3}=6-\frac{8}{3}=\frac{10}{3}$. At $y=-1$: $\frac{1}{2}-2+\frac{1}{3}=-\frac{7}{6}$.

$$A=\frac{10}{3}-\left(-\frac{7}{6}\right)=\frac{20}{6}+\frac{7}{6}=\boxed{\frac{27}{6}=\frac{9}{2}}.$$

Attempting this same region as an integral with respect to x would require splitting into two pieces (since $x=y^2$ is not a single function of x on the whole region) — integrating with respect to y avoids that split entirely.

8.4.3 Regions where the curves cross more than once

If $f(x)-g(x)$ changes sign more than once on the interval of interest, the "top" and "bottom" curves swap roles partway through, and a single integral of $f(x)-g(x)$ over the whole interval would let positive and negative pieces cancel — giving the wrong (net signed) area instead of the true enclosed area. The fix is the same idea used for total distance in Section 2: split the interval at *every* crossing point, and add the absolute value of each piece.

Area with multiple crossings

If f and g cross at $x=a=x_0<x_1<x_2<\cdots<x_n=b$, the total enclosed area is

$$A=\sum_{i=1}^{n}\left|\int_{x_{i-1}}^{x_i}\big[f(x)-g(x)\big]\,dx\right|=\int_a^b\big|f(x)-g(x)\big|\,dx.$$

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Khi hai đường cong cắt nhau **nhiều hơn một lần** trong đoạn đang xét, hàm hiệu $f(x)-g(x)$ sẽ đổi dấu tại mỗi giao điểm — nghĩa là "đường trên" và "đường dưới" hoán đổi vai trò cho nhau. Nếu chỉ lấy một tích phân duy nhất trên cả đoạn mà không tách nhỏ, các phần diện tích dương và âm sẽ triệt tiêu lẫn nhau, cho ra **diện tích có dấu (net area)** thay vì tổng diện tích thực sự bị bao bởi hai đường. Cách xử lý đúng: tìm *tất cả* giao điểm, tách tích phân tại từng giao điểm, lấy trị tuyệt đối của mỗi mảnh rồi cộng lại — hoàn toàn giống logic tính quãng đường đi được từ vận tốc đổi dấu.

Ví dụ mẫu · Two curves that cross three times

Find the total area of the region(s) enclosed between $f(x)=x^3-3x$ and $g(x)=x$.

Step 1 — find all crossings: $x^3-3x=x\Rightarrow x^3-4x=0\Rightarrow x(x^2-4)=0\Rightarrow x=-2,\,0,\,2$. Three crossing points, so *two* separate regions.

Step 2 — determine top/bottom on each piece. Let $h(x)=f(x)-g(x)=x^3-4x$. Test $x=-1$: $h(-1)=-1+4=3>0$, so $f>g$ on $(-2,0)$. Test $x=1$: $h(1)=1-4=-3<0$, so $g>f$ on $(0,2)$.

Step 3 — integrate each piece separately. An antiderivative of $h(x)=x^3-4x$ is $H(x)=\frac{x^4}{4}-2x^2$, with $H(-2)=4-8=-4$, $H(0)=0$, $H(2)=4-8=-4$.

$$\int_{-2}^{0}(f-g)\,dx=H(0)-H(-2)=0-(-4)=4,$$

$$\int_{0}^{2}(g-f)\,dx=-\big[H(2)-H(0)\big]=-(-4-0)=4.$$

Step 4 — total area:

$$A = 4 + 4 = \boxed{8}.$$

8.5 Volumes with Known Cross-Sections

8.5.1 Squares and rectangles

Some solids are not solids of revolution at all: instead, we are told the shape of every **cross-section** perpendicular to a given axis, and that shape's size is built from the distance between two bounding curves. If $A(x)$ denotes the area of the cross-section at position x, slicing the solid into thin slabs of thickness dx and summing their volumes gives:

Volume by Known Cross-Sections

$$V = \int_a^b A(x)\,dx.$$

If the cross-sections perpendicular to the x-axis are **squares** whose side length equals the distance $s(x)$ between two bounding curves, then $A(x) = [s(x)]^2$. If the cross-sections are **rectangles** whose height is some fixed multiple k of the base $s(x)$ (e.g. "height twice the base"), then $A(x) = k\,[s(x)]^2$.

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Với bài toán **thiết diện đã biết (known cross-sections)**, khối không phải là khối tròn xoay — thay vào đó, mỗi lát cắt vuông góc với một trục có hình dạng cho trước (vuông, chữ nhật, tam giác, nửa hình tròn...) với kích thước xây dựng từ **khoảng cách giữa hai đường cong biên**. Luôn bắt đầu bằng cách xác định độ dài cạnh đáy $s(x)$ (khoảng cách giữa hai đường), sau đó suy ra diện tích thiết diện $A(x)$ theo đúng công thức hình học của hình đó, rồi lấy tích phân $V = \int_a^b A(x)\,dx$ — hoàn toàn không có π trong công thức trừ khi hình thiết diện là hình tròn hoặc nửa hình tròn.

Ví dụ mẫu · Square cross-sections

The base of a solid is the region between $y = \sqrt{x}$ and $y = x^2$ on $[0, 1]$. Cross-sections perpendicular to the x-axis are squares whose side length equals the distance between the two curves. Find the volume of the solid.

Side length: $s(x) = \sqrt{x} - x^2$ (valid since $\sqrt{x} \geq x^2$ on $[0, 1]$).

Cross-sectional area: $A(x) = \left(\sqrt{x} - x^2\right)^2 = x - 2x^{5/2} + x^4$.

Volume:

$$V = \int_0^1 \left(x - 2x^{5/2} + x^4\right) dx = \left[\frac{x^2}{2} - \frac{4}{7}x^{7/2} + \frac{x^5}{5}\right]_0^1 = \frac{1}{2} - \frac{4}{7} + \frac{1}{5}.$$

Common denominator 70: $\frac{35}{70} - \frac{40}{70} + \frac{14}{70} = \boxed{\frac{9}{70}}$.

8.5.2 Equilateral triangles and semicircles

The same method applies to any cross-sectional shape, as long as we can express that shape's area in terms of a single length taken from the bounding curves. Two shapes appear constantly on the AP exam:

Deriving the area formulas

Equilateral triangle with side length s: drop an altitude to split it into two $30°$–$60°$–$90°$ triangles, giving height $h = \frac{\sqrt{3}}{2}s$, so

$$A_{\triangle} = \frac{1}{2} \cdot s \cdot \frac{\sqrt{3}}{2}s = \frac{\sqrt{3}}{4}s^2.$$

Semicircle whose *diameter* equals the base length s (so radius $r = \frac{s}{2}$):

$$A_{\text{semi}} = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi\left(\frac{s}{2}\right)^2 = \frac{\pi}{8}s^2.$$

Cross-section shape	Governing length	Area formula $A(x)$
Square	side $s(x)$	$[s(x)]^2$
Rectangle, height $= k\times$ base	base $s(x)$	$k\,[s(x)]^2$
Equilateral triangle	side $s(x)$	$\frac{\sqrt{3}}{4}\,[s(x)]^2$
Semicircle (diameter $= s(x)$)	diameter $s(x)$	$\frac{\pi}{8}\,[s(x)]^2$

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Với **tam giác đều** cạnh s, hạ đường cao chia thành hai tam giác vuông 30–60–90 để suy ra chiều cao $h = \frac{\sqrt{3}}{2}s$, từ đó $A_{\triangle} = \frac{\sqrt{3}}{4}s^2$. Với **nửa hình tròn**, điểm dễ nhầm nhất là: độ dài $s(x)$ lấy được từ hai đường cong biên thường đóng vai trò là **đường kính** (không phải bán kính!) của nửa hình tròn, nên bán kính thực sự là $r = s/2$, và diện tích nửa hình tròn là $A_{\text{semi}} = \frac{1}{2}\pi r^2 = \frac{\pi}{8}s^2$ — chứ không phải $\frac{1}{2}\pi s^2$.

Ví dụ mẫu · Equilateral-triangle cross-sections

The base of a solid is the region under $y = \sqrt{x}$ from $x = 0$ to $x = 4$ (bounded below by the x-axis). Cross-sections perpendicular to the x-axis are equilateral triangles with side length equal to $\sqrt{x}$. Find the volume.

$$A(x) = \frac{\sqrt{3}}{4}\left(\sqrt{x}\right)^2 = \frac{\sqrt{3}}{4}x, \qquad V = \int_0^4 \frac{\sqrt{3}}{4}x\,dx = \frac{\sqrt{3}}{4}\left[\frac{x^2}{2}\right]_0^4 = \frac{\sqrt{3}}{4}\cdot 8 = \boxed{2\sqrt{3}}.$$

Ví dụ mẫu · Semicircular cross-sections

The base of a solid is the region under $y = 4 - x^2$ from $x = -2$ to $x = 2$ (where the parabola meets the x-axis). Cross-sections perpendicular to the x-axis are semicircles whose diameter equals the height of the base region, $4 - x^2$. Find the volume.

Radius: $r(x) = \dfrac{4 - x^2}{2}$, so $A(x) = \dfrac{\pi}{8}(4 - x^2)^2$.

Expand: $(4 - x^2)^2 = 16 - 8x^2 + x^4$.

$$V = \frac{\pi}{8}\int_{-2}^{2}(16 - 8x^2 + x^4)\,dx = \frac{\pi}{8}\left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_{-2}^{2}.$$

By symmetry, evaluate at $x = 2$ and double: $16(2) - \frac{8(8)}{3} + \frac{32}{5} = 32 - \frac{64}{3} + \frac{32}{5}$. Common

denominator 15: $\frac{480}{15} - \frac{320}{15} + \frac{96}{15} = \frac{256}{15}$; doubling gives $\frac{512}{15}$.

$$V = \frac{\pi}{8} \cdot \frac{512}{15} = \boxed{\frac{64\pi}{15}}.$$

8.6 The Disk Method

8.6.1 Volumes of revolution by disks

When a plane region is revolved 360° about a line (the **axis of revolution**), it sweeps out a three-dimensional solid. If the region is bounded on one side by that axis (no gap between the region and the axis), each cross-section perpendicular to the axis is a solid circular **disk**, and the known-cross-sections formula with $A(x) = \pi[R(x)]^2$ becomes:

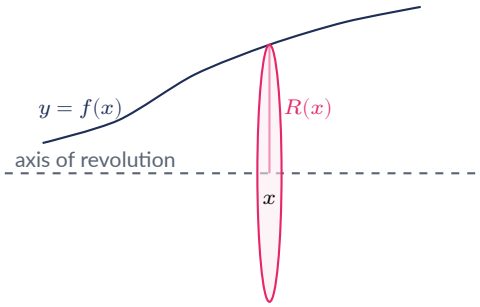
Disk Method

$$V = \pi \int_a^b [R(x)]^2 \, dx,$$

where $R(x)$ is the **radius** of the disk at position x — the perpendicular distance from the axis of revolution to the bounding curve. When revolving about the x-axis, $R(x) = f(x)$ (the curve's own y-value). When revolving about the y-axis (integrating in y), $R(y)$ is the curve's x-value. When revolving about **any other horizontal or vertical line** $y = k$ or $x = k$, the radius must be recomputed as the *distance from that line* to the curve: $R(x) = f(x) - k$ (or $k - f(x)$, whichever is positive) for a horizontal axis $y = k$, and analogously for a vertical axis $x = k$.

VN **GIẢI THÍCH TIẾNG VIỆT**

Phương pháp **đĩa (disk)** áp dụng khi miền được quay quanh một trục và **chạm trực tiếp** vào trục đó (không có khoảng hở) — mỗi lát cắt vuông góc với trục là một hình tròn đặc, bán kính $R(x)$ chính là khoảng cách từ trục quay đến đường cong biên. Khi quay quanh trục hoành, $R(x)$ đơn giản là $f(x)$; khi quay quanh trục tung, $R(y)$ là giá trị x của đường cong. Nhưng khi quay quanh một đường thẳng **khác** (ví dụ $y = k$ hoặc $x = k$), bắt buộc phải **tính lại bán kính là khoảng cách đến đường đó** — không được dùng thẳng giá trị $f(x)$ như thể trục vẫn là trục hoành.



Side view of a solid of revolution about the axis (dashed): the vertical segment of length $R(x)$ sweeps out the circular disk (shown edge-on as a thin ellipse) at position x.

Ví dụ mẫu · Disk method about the x-axis

The region under $y = \sqrt{x}$ from $x = 0$ to $x = 4$ is revolved about the x-axis. Find the volume of the resulting solid.

Since the region touches the x-axis directly, $R(x) = \sqrt{x}$, so $[R(x)]^2 = x$.

$$V = \pi \int_0^4 x\,dx = \pi \left[\frac{x^2}{2}\right]_0^4 = \pi \cdot \frac{16}{2} = \boxed{8\pi}.$$

Ví dụ mẫu · Disk method about the y-axis

The region bounded by $x = y^2$, the y-axis, and $y = 2$ is revolved about the y-axis. Find the volume.

Here we integrate with respect to y, and since the region touches the y-axis directly, $R(y) = y^2$, so $[R(y)]^2 = y^4$.

$$V = \pi \int_0^2 y^4\,dy = \pi \left[\frac{y^5}{5}\right]_0^2 = \pi \cdot \frac{32}{5} = \boxed{\frac{32\pi}{5}}.$$

Ví dụ mẫu · Disk method about a line other than an axis

The region bounded above by $y = \sqrt{x}$, below by the line $y = -1$, and on the right by $x = 4$ is revolved about the line $y = -1$. Find the volume.

The region's lower boundary is exactly the axis of revolution $y = -1$, so this is a genuine disk (no gap). The radius is the **vertical distance from $y = -1$ up to the curve**:

$$R(x) = \sqrt{x} - (-1) = \sqrt{x} + 1.$$

$$[R(x)]^2 = \left(\sqrt{x} + 1\right)^2 = x + 2\sqrt{x} + 1.$$

$$V = \pi \int_0^4 \left(x + 2\sqrt{x} + 1\right) dx = \pi \left[\frac{x^2}{2} + \frac{4}{3}x^{3/2} + x\right]_0^4.$$

At $x = 4$: $\frac{16}{2} + \frac{4}{3}(8) + 4 = 8 + \frac{32}{3} + 4 = 12 + \frac{32}{3} = \frac{36}{3} + \frac{32}{3} = \frac{68}{3}$.

$$V = \boxed{\frac{68\pi}{3}}.$$

Note how different this is from revolving the same curve about the x-axis: shifting the axis down to $y = -1$ makes *every* radius 1 unit longer, and because the radius gets squared, that shift changes the volume by far more than a simple constant.

(!) Lỗi thường gặp: Khi trục quay **không phải** là trục hoành hay trục tung, bán kính $R(x)$ **không còn bằng** $f(x)$ nữa — đây là lỗi sai bán kính phổ biến nhất trong các bài toán thể tích. Bán kính luôn phải là **khoảng cách từ trục quay đến đường cong**: nếu quay quanh $y = k$ và đường cong nằm phía trên k, thì $R(x) = f(x) - k$; nếu đường cong nằm phía dưới k, thì $R(x) = k - f(x)$. Tương tự với trục dọc $x = k$: $R(y) = g(y) - k$ hoặc $k - g(y)$ tuỳ vị trí. Luôn vẽ (hoặc hình dung) một lát cắt và đo khoảng cách thực tế từ trục đến đường cong trước khi viết công thức — tuyệt đối không mặc định dùng $f(x)$ hay $g(y)$ một cách máy móc.

8.7 The Washer Method

8.7.1 Volumes of revolution with a gap

If the region being revolved does *not* touch the axis of revolution — there is a gap between the axis and the nearer boundary curve — each cross-section is not a solid disk but an **annulus** (a washer): a disk with a smaller disk removed from its center.

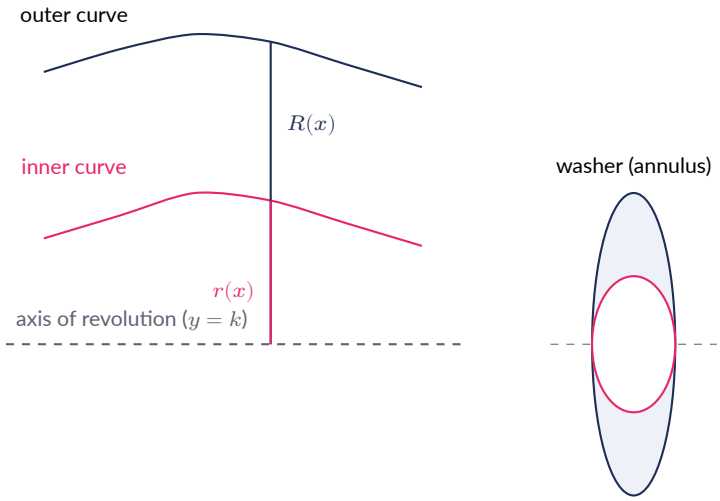
Washer Method

$$V = \pi \int_a^b \left([R(x)]^2 - [r(x)]^2 \right) dx,$$

where $R(x)$ is the **outer radius** (distance from the axis to the *farther* bounding curve) and $r(x)$ is the **inner radius** (distance from the axis to the *nearer* bounding curve). As with the disk method, if the axis of revolution is not the x-axis or y-axis, both $R(x)$ and $r(x)$ must be recomputed as distances to that specific line.

VN **GIẢI THÍCH TIẾNG VIỆT**

Phương pháp **vành khăn (washer)** dùng khi miền được quay **không chạm** vào trục quay — luôn có một "lỗ hổng" ở giữa. Mỗi lát cắt lúc này là một hình vành khăn (đĩa lớn trừ đĩa nhỏ), với $R(x)$ là bán kính **ngoài** (khoảng cách từ trục đến đường cong *xa* trục hơn) và $r(x)$ là bán kính **trong** (khoảng cách đến đường cong *gần* trục hơn). Thể tích luôn là $V = \pi \int_a^b \left([R(x)]^2 - [r(x)]^2 \right) dx$ — bình phương **từng bán kính riêng biệt** rồi mới trừ, tuyệt đối không được trừ hai bán kính trước rồi mới bình phương.



Side view: outer radius $R(x)$ reaches the farther curve, inner radius $r(x)$ reaches the nearer curve. The washer (shown edge-on, right) is a disk of radius $R(x)$ with a hole of radius $r(x)$ removed.

Ví dụ mẫu · Washer method about the x-axis

The region between $y = \sqrt{x}$ and $y = x^2$ on $[0, 1]$ is revolved about the x-axis. Find the volume.

On $[0, 1]$, $\sqrt{x} \geq x^2$, so the outer radius is $R(x) = \sqrt{x}$ (farther from the x-axis) and the inner radius is $r(x) = x^2$ (nearer, since the region does not include the strip between $y = 0$ and $y = x^2$).

$$V = \pi \int_0^1 \left[(\sqrt{x})^2 - (x^2)^2 \right] dx = \pi \int_0^1 (x - x^4)\, dx = \pi \left[\frac{x^2}{2} - \frac{x^5}{5} \right]_0^1 = \pi \left(\frac{1}{2} - \frac{1}{5} \right) = \pi \cdot \frac{3}{10} = \boxed{\frac{3\pi}{10}}.$$

Ví dụ mẫu · Washer method about a line other than an axis

The region bounded by $y = x^2$ and $y = 2x$ is revolved about the line $y = 5$. Find the volume.

Intersections: $x^2 = 2x \Rightarrow x(x-2) = 0 \Rightarrow x = 0,\ 2$.

Radii, measured from $y = 5$: on $(0, 2)$, $2x \geq x^2$ (test $x = 1$: $2 > 1$), so $y = 2x$ is the *upper* curve, closer to the axis $y = 5$; $y = x^2$ is the *lower* curve, farther from the axis. Since distance is measured *down* from $y = 5$, the farther (lower) curve gives the larger radius:

$$R(x) = 5 - x^2 \quad \text{(outer, to the lower curve)}, \qquad r(x) = 5 - 2x \quad \text{(inner, to the upper curve)}.$$

$$[R(x)]^2 - [r(x)]^2 = (5-x^2)^2 - (5-2x)^2 = (25 - 10x^2 + x^4) - (25 - 20x + 4x^2) = x^4 - 14x^2 + 20x.$$

$$V = \pi \int_0^2 (x^4 - 14x^2 + 20x)\, dx = \pi \left[\frac{x^5}{5} - \frac{14x^3}{3} + 10x^2 \right]_0^2.$$

At $x = 2$: $\frac{32}{5} - \frac{112}{3} + 40$. Common denominator 15: $\frac{96}{15} - \frac{560}{15} + \frac{600}{15} = \frac{136}{15}$.

$$V = \boxed{\frac{136\pi}{15}}.$$

(!) Lỗi thường gặp: Sai lầm kinh điển trong phương pháp vành khăn: viết $(R(x) - r(x))^2$ thay vì $[R(x)]^2 - [r(x)]^2$. Hai biểu thức này **hoàn toàn khác nhau** về mặt đại số — ví dụ nếu $R = 5, r = 3$ thì $[R]^2 - [r]^2 = 25 - 9 = 16$, trong khi $(R - r)^2 = 2^2 = 4$. Luôn **bình phương từng bán kính một cách riêng biệt trước**, rồi mới lấy hiệu: $\pi \int ([R(x)]^2 - [r(x)]^2)\, dx$. Ngoài ra, khi trục quay không phải trục hoành/trục tung, hãy luôn kiểm tra lại xem đường cong nào cho bán kính **lớn hơn** (bán kính ngoài R) và đường nào cho bán kính **nhỏ hơn** (bán kính trong r) — thứ tự này có thể đảo ngược so với thứ tự "trên/dưới" quen thuộc, tuỳ vào vị trí của trục quay.

8.7.2 Summary: choosing a volume method

Before starting any volume problem, decide which of the three formulas actually applies. All three are built from the same idea — slice, find the cross-sectional area $A(x)$, integrate — but the shape and the presence of π differ:

Method	When to use	Formula
Known cross-sections	Non-circular slabs (square, rectangle, triangle, semicircle) built from the gap between two curves	$V = \int_a^b A(x)\, dx$, with $A(x)$ from the shape's own area formula
Disk	Region touches the axis of revolution directly (no gap)	$V = \pi \int_a^b [R(x)]^2\, dx$
Washer	Region has a gap between it and the axis of revolution	$V = \pi \int_a^b ([R(x)]^2 - [r(x)]^2)\, dx$

VN GIẢI THÍCH TIẾNG VIỆT

Trước khi bắt tay vào bài toán thể tích, hãy tự hỏi: (1) Thiết diện có phải hình tròn/nửa hình tròn không? Nếu không (vuông, chữ nhật, tam giác) thì **không có** π trong công thức. (2) Nếu là hình tròn (do quay quanh một trục), miền có **chạm trực tiếp** vào trục quay không? Nếu có $\Rightarrow$ dùng **đĩa**; nếu có khoảng hở $\Rightarrow$ dùng **vành khăn**. Trả lời đúng hai câu hỏi này trước khi viết bất kỳ công thức nào sẽ tránh được phần lớn lỗi sai trong chương này.

8.8 The Arc Length of a Smooth, Planar Curve and Distance Traveled

Phần dành riêng cho AP Calculus BC (CED 8.13): độ dài cung của đồ thị $y = f(x)$ hoặc $x = g(y)$, cách dựng công thức từ tổng độ dài các đoạn thẳng, các bài tính ra dạng đóng (biểu thức dưới căn là bình phương đúng), bài dùng máy tính (làm tròn 3 chữ số thập phân), và phân biệt độ dài cung với quãng đường đi được của vật chuyển động thẳng.

Every integral in this unit so far has measured an *area* or a *volume* (or an accumulated amount). The last application measures a *length*: how long is the graph of $y = f(x)$ between $x = a$ and $x = b$, if it were straightened out like a piece of string?

8.8.1 From line segments to an integral

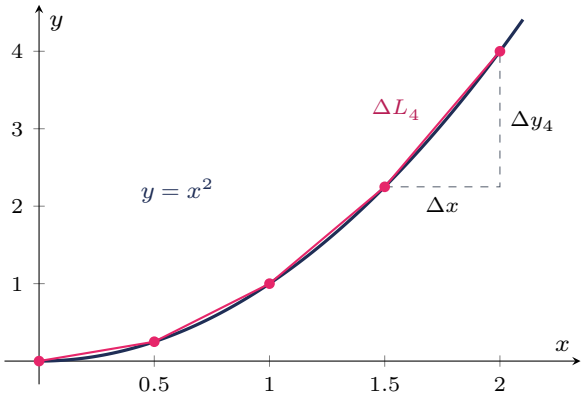
Partition $[a, b]$ into n subintervals of width Δx and join consecutive points $(x_i, f(x_i))$ on the graph by straight segments. By the Pythagorean theorem, the i-th segment has length

$$\Delta L_i = \sqrt{(\Delta x)^2 + (\Delta y_i)^2} = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}\,\Delta x.$$

If f is differentiable, the Mean Value Theorem gives, in each subinterval, a point c_i where $f'(c_i) = \Delta y_i/\Delta x$. So the length of the whole polygonal path is the Riemann sum

$$\sum_{i=1}^{n} \sqrt{1 + [f'(c_i)]^2}\,\Delta x,$$

and as $n \to \infty$ it converges to a definite integral.



Segments n	Polygon length
1	$\sqrt{20} \approx 4.472$
2	$\sqrt{2} + \sqrt{10} \approx 4.576$
4	≈ 4.627
$\to \infty$	$\displaystyle\int_0^2 \sqrt{1 + 4x^2}\,dx \approx 4.647$

A chord is never longer than the arc it spans, so every polygon *underestimates* the length; refining the partition increases the polygon length toward the arc length.

Arc length

If f' is continuous on $[a, b]$ (the curve is **smooth**), the length of the graph of $y = f(x)$ from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

If the curve is given as $x = g(y)$ with g' continuous on $[c, d]$, its length from $y = c$ to $y = d$ is

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

Both come from the same "differential triangle" $ds = \sqrt{(dx)^2 + (dy)^2}$.

VN **GIẢI THÍCH TIẾNG VIỆT**

Độ dài cung (arc length) của đồ thị $y = f(x)$ trên $[a, b]$ là $L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$. Công thức đến từ việc xấp xỉ đường cong bằng các đoạn thẳng nhỏ: mỗi đoạn dài $\sqrt{(\Delta x)^2 + (\Delta y)^2}$ (định lý Pythagore), rồi lấy giới hạn. Nếu đường cong cho dưới dạng $x = g(y)$ thì đổi vai trò: $L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$ — hữu ích khi $\frac{dy}{dx}$ không xác định (tiếp tuyến thẳng đứng) tại một đầu mút. Hầu hết bài độ dài cung trên đề AP là **câu dùng máy tính**: chỉ cần thiết lập đúng tích phân và cho kết quả 3 chữ số thập phân. Bài tính tay chỉ xuất hiện khi $1 + [f'(x)]^2$ là một **bình phương đúng**.

8.8.2 Arc lengths with a closed form

The square root makes most arc-length integrals impossible to evaluate by hand. The exceptions are carefully chosen functions for which $1 + [f'(x)]^2$ is a perfect square, or simplifies to something easy to integrate.

Ví dụ mẫu · A power function

Find the length of the curve $y = \frac{2}{3}x^{3/2}$ from $x = 0$ to $x = 3$.

$\frac{dy}{dx} = \frac{2}{3} \cdot \frac{3}{2}x^{1/2} = x^{1/2}$, so $1 + \left(\frac{dy}{dx}\right)^2 = 1 + x$. Then

$$L = \int_0^3 \sqrt{1 + x}\, dx = \left[\tfrac{2}{3}(1 + x)^{3/2}\right]_0^3 = \tfrac{2}{3}(4^{3/2} - 1^{3/2}) = \tfrac{2}{3}(8 - 1) = \frac{14}{3} \approx 4.667.$$

Ví dụ mẫu · A perfect square under the root

Find the length of the curve $y = \frac{x^2}{8} - \ln x$ for $1 \le x \le 4$.

$\frac{dy}{dx} = \frac{x}{4} - \frac{1}{x}$. Square it: $\left(\frac{x}{4} - \frac{1}{x}\right)^2 = \frac{x^2}{16} - \frac{1}{2} + \frac{1}{x^2}$. Adding 1 changes only the middle sign:

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{x^2}{16} + \frac{1}{2} + \frac{1}{x^2} = \left(\frac{x}{4} + \frac{1}{x}\right)^2.$$

Since $\frac{x}{4} + \frac{1}{x} > 0$ on $[1, 4]$, the square root is simply $\frac{x}{4} + \frac{1}{x}$:

$$L = \int_1^4 \left(\frac{x}{4} + \frac{1}{x}\right)dx = \left[\frac{x^2}{8} + \ln x\right]_1^4 = \left(2 + \ln 4\right) - \left(\frac{1}{8} + 0\right) = \frac{15}{8} + \ln 4 \approx 3.261.$$

The perfect-square pattern. If $f'(x) = A - B$ where $AB = \frac{1}{4}$, then

$$1 + [f'(x)]^2 = 1 + A^2 - 2AB + B^2 = A^2 + 2AB + B^2 = (A + B)^2.$$

In the example above, $A = \frac{x}{4}$ and $B = \frac{1}{x}$ (so $AB = \frac{1}{4}$). The same pattern drives $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$, with $A = \dfrac{x^2}{2}$, $B = \dfrac{1}{2x^2}$.

Ví dụ mẫu · The pattern again

Find the length of $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$ on $[1, 2]$.

$\dfrac{dy}{dx} = \dfrac{x^2}{2} - \dfrac{1}{2x^2}$, and $\dfrac{x^2}{2} \cdot \dfrac{1}{2x^2} = \dfrac{1}{4}$, so $1 + \left(\dfrac{dy}{dx}\right)^2 = \left(\dfrac{x^2}{2} + \dfrac{1}{2x^2}\right)^2$. Hence

$$L = \int_1^2 \left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx = \left[\frac{x^3}{6} - \frac{1}{2x}\right]_1^2 = \left(\frac{8}{6} - \frac{1}{4}\right) - \left(\frac{1}{6} - \frac{1}{2}\right) = \frac{13}{12} + \frac{1}{3} = \frac{17}{12} \approx 1.417.$$

8.8.3 Arc length with a calculator

For almost every other function, the arc-length integral has no elementary antiderivative. On the calculator-active part of the exam, the task is to **set up** the integral correctly and evaluate it numerically, reporting three decimal places.

Ví dụ mẫu · Calculator-active arc length

(a) Find the length of the parabola $y = x^2$ from $x = 0$ to $x = 2$.
(b) Find the length of one arch of $y = \sin x$, $0 \le x \le \pi$.
(c) A suspension cable hangs in the shape $y = 0.02x^2$ (in meters) between towers at $x = -50$ and $x = 50$. How long is the cable?

(a) $\dfrac{dy}{dx} = 2x$: $L = \displaystyle\int_0^2 \sqrt{1 + 4x^2}\, dx \approx 4.647$.

(b) $\dfrac{dy}{dx} = \cos x$: $L = \displaystyle\int_0^{\pi} \sqrt{1 + \cos^2 x}\, dx \approx 3.820$.

(c) $\dfrac{dy}{dx} = 0.04x$: $L = \displaystyle\int_{-50}^{50} \sqrt{1 + 0.0016x^2}\, dx \approx 147.894$ meters. (By symmetry this also equals $2\int_0^{50} \sqrt{1 + 0.0016x^2}\, dx$.)

On a free-response question, write the integral *with its limits and integrand* before the numerical answer — the setup earns a point by itself, and a bare decimal earns nothing if it is wrong.

8.8.4 Curves given as $x = g(y)$

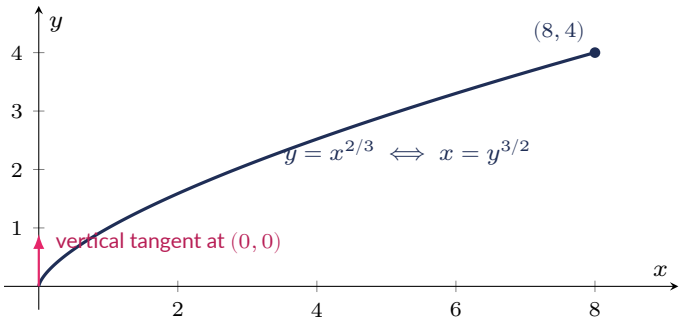
Integrating with respect to y is useful in two situations: when the curve is naturally described as $x = g(y)$, and when $\frac{dy}{dx}$ is undefined at an endpoint (a vertical tangent), making $\sqrt{1 + [f'(x)]^2}$ unbounded there. Viewed sideways, the same curve usually has a horizontal tangent at that point and a perfectly continuous $\frac{dx}{dy}$.

Ví dụ mẫu · Integrating in y

Find the length of the curve $x = \frac{1}{3}(y^2+2)^{3/2}$ from $y=0$ to $y=1$.

$\frac{dx}{dy} = \frac{1}{3}\cdot\frac{3}{2}(y^2+2)^{1/2}\cdot 2y = y\sqrt{y^2+2}$, so

$$1+\left(\frac{dx}{dy}\right)^2 = 1+y^2(y^2+2) = y^4+2y^2+1 = (y^2+1)^2, \qquad L=\int_0^1 (y^2+1)\,dy = \frac{1}{3}+1 = \frac{4}{3}.$$



The curve $y = x^{2/3}$ leaves the origin vertically, so $\frac{dy}{dx} = \frac{2}{3}x^{-1/3}$ is unbounded at $x=0$. Written as $x = y^{3/2}$, the derivative $\frac{dx}{dy} = \frac{3}{2}y^{1/2}$ is continuous on $[0,4]$.

Ví dụ mẫu · Avoiding a vertical tangent

Find the length of $y = x^{2/3}$ from $(0,0)$ to $(8,4)$.

In terms of x, $\frac{dy}{dx} = \frac{2}{3x^{1/3}}$ is undefined at $x=0$, so $\int_0^8 \sqrt{1+\frac{4}{9x^{2/3}}}\,dx$ would be improper.

Instead write $x = y^{3/2}$ for $0\le y\le 4$: $\frac{dx}{dy} = \frac{3}{2}y^{1/2}$, and

$$L=\int_0^4 \sqrt{1+\frac{9}{4}y}\,dy = \frac{4}{9}\cdot\frac{2}{3}\left[\left(1+\frac{9}{4}y\right)^{3/2}\right]_0^4 = \frac{8}{27}(10^{3/2}-1) = \frac{8}{27}(10\sqrt{10}-1) \approx 9.073.$$

8.8.5 Arc length versus distance traveled

The phrase "distance traveled" belongs to motion, and in Unit 8 motion is *along a line*. For a particle moving on the x-axis with velocity $v(t)$, the total distance traveled on $[t_1,t_2]$ is $\int_{t_1}^{t_2}|v(t)|\,dt$, as developed in the section on motion along a line. The arc length of the *graph* of its position function $x(t)$ is a different number with no physical meaning: the graph lives in the (t,x)-plane, where one axis is time, so "length" there mixes seconds with meters.

Ví dụ mẫu · Two different numbers

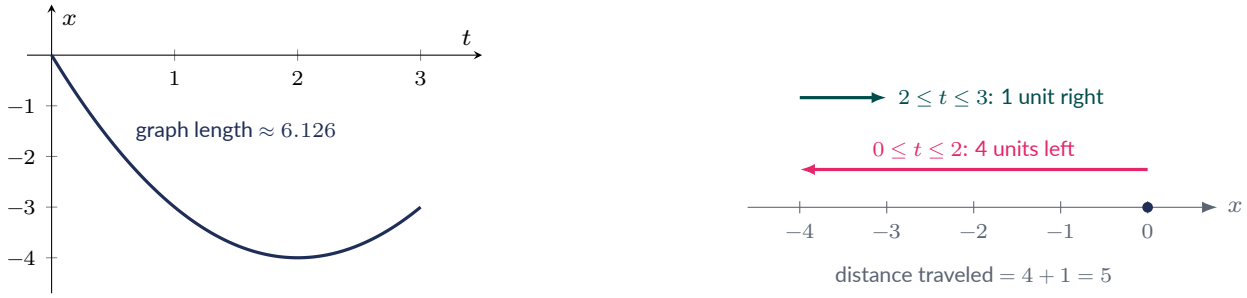
A particle moves along the x-axis with position $x(t) = t^2 - 4t$ for $0\le t\le 3$.
(a) Find the total distance the particle travels. **(b)** Compare with the arc length of the graph of $x(t)$.

(a) $v(t) = 2t-4$, which is negative on $[0,2)$ and positive on $(2,3]$. The particle moves left from

$x = 0$ to $x(2) = -4$, then right to $x(3) = -3$:

$$\int_0^3 |2t-4|\,dt = \int_0^2 (4-2t)\,dt + \int_2^3 (2t-4)\,dt = 4 + 1 = 5.$$

(b) The graph of $x = t^2 - 4t$ in the tx-plane has length $\displaystyle\int_0^3 \sqrt{1+(2t-4)^2}\,dt \approx 6.126$. This exceeds 5 because it also "counts" the passage of time along the horizontal axis; it is **not** a distance the particle travels.



Left: the position graph $x = t^2 - 4t$ (axes are time and position). Right: the actual motion on the x-axis. Only the right-hand picture measures how far the particle goes.

Which length formula?

Situation	Formula	Where in the course
Length of the graph $y = f(x)$, $a \le x \le b$	$\displaystyle\int_a^b \sqrt{1+[f'(x)]^2}\,dx$	Topic 8.13 (this section)
Distance traveled along a line, velocity $v(t)$	$\displaystyle\int_{t_1}^{t_2} \lvert v(t)\rvert\,dt$	Topic 8.2
Distance traveled in the plane, $x(t)$ and $y(t)$	$\displaystyle\int_{t_1}^{t_2} \sqrt{x'(t)^2+y'(t)^2}\,dt$	Unit 9 (parametric)

In Unit 9 the two ideas merge: for a particle moving in the plane, the distance it travels *is* the arc length of its path (when the path is traced once).

(!) Lỗi thường gặp: Common errors: (1) writing $\sqrt{1+f'(x)}$ or $1+[f'(x)]^2$ — the derivative must be squared and the whole sum must be under the root; (2) simplifying $\sqrt{1+[f'(x)]^2}$ to $1+f'(x)$, which is false (a square root does not distribute over a sum); (3) using arc length of the position graph for "total distance traveled" by a particle on a line; (4) giving fewer than three correct decimal places on a calculator-active answer.

8.9 BC Practice: Arc Length

Bài tập luyện tập

Which integral gives the length of the curve $y = e^{2x}$ from $x = 0$ to $x = 1$?

(A) $\displaystyle\int_0^1 \sqrt{1 + e^{4x}}\,dx$

(B) $\displaystyle\int_0^1 \sqrt{1 + 2e^{2x}}\,dx$

(C) $\displaystyle\int_0^1 \sqrt{1 + 4e^{4x}}\,dx$

(D) $\displaystyle\int_0^1 (1 + 4e^{4x})\,dx$

Lời giải chi tiết

$\dfrac{dy}{dx} = 2e^{2x}$, so $\left(\dfrac{dy}{dx}\right)^2 = 4e^{4x}$ and $L = \displaystyle\int_0^1 \sqrt{1 + 4e^{4x}}\,dx$ (≈ 6.495). **(C).** (A) squares e^{2x} but forgets the chain-rule factor 2; (B) does not square the derivative; (D) omits the square root.

Bài tập luyện tập

Find the exact length of $y = \ln(\cos x)$ for $0 \le x \le \dfrac{\pi}{4}$.

Lời giải chi tiết

$\dfrac{dy}{dx} = \dfrac{-\sin x}{\cos x} = -\tan x$, so $1 + \tan^2 x = \sec^2 x$ and $\sqrt{\sec^2 x} = \sec x$ (positive on $[0, \frac{\pi}{4}]$).

$$L = \int_0^{\pi/4} \sec x\,dx = \Big[\ln|\sec x + \tan x|\Big]_0^{\pi/4} = \ln(\sqrt{2} + 1) - \ln 1 = \ln(1 + \sqrt{2}) \approx 0.881.$$

Bài tập luyện tập

Find the length of $y = \dfrac{x^2}{4} - \dfrac{\ln x}{2}$ for $1 \le x \le e$.

Lời giải chi tiết

$\dfrac{dy}{dx} = \dfrac{x}{2} - \dfrac{1}{2x}$, and $\dfrac{x}{2} \cdot \dfrac{1}{2x} = \dfrac{1}{4}$, so $1 + \left(\dfrac{dy}{dx}\right)^2 = \left(\dfrac{x}{2} + \dfrac{1}{2x}\right)^2$.

$$L = \int_1^e \left(\frac{x}{2} + \frac{1}{2x}\right)dx = \left[\frac{x^2}{4} + \frac{\ln x}{2}\right]_1^e = \frac{e^2}{4} + \frac{1}{2} - \frac{1}{4} = \frac{e^2 + 1}{4} \approx 2.097.$$

Bài tập luyện tập

The length of the curve $y = \frac{1}{3}(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 3$ is

(A) 9

(B) 12

(C) $\dfrac{40}{3}$

(D) $\sqrt{145}$

Lời giải chi tiết

$\frac{dy}{dx} = x\sqrt{x^2+2}$, so $1 + x^2(x^2+2) = (x^2+1)^2$, and $L = \int_0^3 (x^2+1)\,dx = 9 + 3 = 12$. **(B).** Choice (A) forgets the $+1$ inside the integral.

Bài tập luyện tập

(Calculator) Find the length of $y = x^3$ from $x = 0$ to $x = 1$, and the length of $x = y^2$ from $y = 0$ to $y = 1$.

Lời giải chi tiết

$y = x^3$: $L = \int_0^1 \sqrt{1+9x^4}\,dx \approx 1.548$.

$x = y^2$: $\frac{dx}{dy} = 2y$, $L = \int_0^1 \sqrt{1+4y^2}\,dy \approx 1.479$. (This arc is the reflection of $y = x^2$, $0 \le x \le 1$, in the line $y = x$, so the two lengths are equal.)

Bài tập luyện tập

Use the two line segments joining the points of $y = x^2$ at $x = 0, 1, 2$ to approximate the length of the curve on $[0, 2]$. Is this an overestimate or an underestimate?

(A) 4.576, underestimate

(B) 4.576, overestimate

(C) 4.472, underestimate

(D) 4.647, exact

Lời giải chi tiết

Points: $(0,0)$, $(1,1)$, $(2,4)$. Segment lengths: $\sqrt{1^2+1^2} = \sqrt{2}$ and $\sqrt{1^2+3^2} = \sqrt{10}$. Total $\sqrt{2} + \sqrt{10} \approx 4.576$. A straight segment is the shortest path between its endpoints, so each chord is shorter than its arc: an **underestimate** (true length ≈ 4.647). **(A).** Choice (C) uses one segment only.

Bài tập luyện tập

A curve $y = f(x)$ passes through $(0, 1)$, and its arc length from $x = 0$ to any $x > 0$ is $s(x) = \int_0^x \sqrt{1+9t^4}\,dt$. If f is increasing, find $f(x)$.

Lời giải chi tiết

Comparing with $s(x) = \int_0^x \sqrt{1+[f'(t)]^2}\,dt$: $[f'(t)]^2 = 9t^4$, so $f'(t) = \pm 3t^2$. Since f is increasing, $f'(x) = 3x^2$, so $f(x) = x^3 + C$; $f(0) = 1$ gives $f(x) = x^3 + 1$.

Bài tập luyện tập

The arc length of the quarter circle $y = \sqrt{4-x^2}$, $0 \le x \le 2$, is known to be π. Show this using the arc length formula, and explain why the integral is improper.

Lời giải chi tiết

$\frac{dy}{dx} = \frac{-x}{\sqrt{4-x^2}}$, so $1 + \frac{x^2}{4-x^2} = \frac{4}{4-x^2}$ and $L = \int_0^2 \frac{2}{\sqrt{4-x^2}}\,dx$. The integrand is unbounded as $x \to 2^-$ (vertical tangent at $(2,0)$), so the integral is improper:

$$L = \lim_{b\to 2^-} \left[2\arcsin\frac{x}{2}\right]_0^b = 2\arcsin 1 - 0 = 2\cdot\frac{\pi}{2} = \pi.$$

Bài tập luyện tập

A particle moves along the x-axis with velocity $v(t) = t^2 - 2t$ for $0 \le t \le 3$. Find the total distance traveled.

(A) 0

(B) $\frac{4}{3}$

(C) $\frac{8}{3}$

(D) $\int_0^3 \sqrt{1 + (t^2-2t)^2}\,dt$

Lời giải chi tiết

$v(t) = t(t-2)$ is negative on $(0,2)$ and positive on $(2,3)$. $\int_0^2 (2t - t^2)\,dt = 4 - \frac{8}{3} = \frac{4}{3}$ and $\int_2^3 (t^2-2t)\,dt = 0 - \left(\frac{8}{3} - 4\right) = \frac{4}{3}$. Distance $= \frac{4}{3} + \frac{4}{3} = \frac{8}{3}$. **(C)**. Choice (A) is the displacement; (D) is an arc-length integral, which does not measure distance along a line.

Bài tập luyện tập

Find the length of the curve $x = \frac{y^3}{6} + \frac{1}{2y}$ for $1 \le y \le 3$.

Lời giải chi tiết

$\frac{dx}{dy} = \frac{y^2}{2} - \frac{1}{2y^2}$; the product of the two terms is $\frac{1}{4}$, so $1 + \left(\frac{dx}{dy}\right)^2 = \left(\frac{y^2}{2} + \frac{1}{2y^2}\right)^2$.

$$L = \int_1^3 \left(\frac{y^2}{2} + \frac{1}{2y^2}\right)dy = \left[\frac{y^3}{6} - \frac{1}{2y}\right]_1^3 = \left(\frac{9}{2} - \frac{1}{6}\right) - \left(\frac{1}{6} - \frac{1}{2}\right) = \frac{14}{3} \approx 4.667.$$

Bài tập luyện tập

Let R be the region bounded by $y = \sqrt{x}$, the x-axis, and $x = 4$.
(a) (Calculator) Find the perimeter of R.
(b) Explain why it is convenient to compute the curved part with respect to y.

Lời giải chi tiết

(a) The boundary consists of the segment on the x-axis from $(0,0)$ to $(4,0)$ (length 4), the vertical segment from $(4,0)$ to $(4,2)$ (length 2), and the arc of $y = \sqrt{x}$ from $(0,0)$ to $(4,2)$. As

$x = y^2$, $0 \le y \le 2$: arc length $= \int_0^2 \sqrt{1 + 4y^2}\, dy \approx 4.647$. Perimeter $\approx 4 + 2 + 4.647 = 10.647$.

(b) With respect to x, $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$ is unbounded at $x = 0$ (vertical tangent), so $\int_0^4 \sqrt{1 + \frac{1}{4x}}\, dx$ is improper. With respect to y, $\frac{dx}{dy} = 2y$ is continuous, and the integral is ordinary — and identical to the length of $y = x^2$ on $[0, 2]$.

8.10 Problem Bank

Bài tập luyện tập

The average value of $f(x) = 3x^2$ on $[0, 2]$ is:

(A) 4 **(C)** 2

(B) 8 **(D)** 6

Lời giải chi tiết

$f_{\text{avg}} = \frac{1}{2-0}\int_0^2 3x^2\, dx = \frac{1}{2}\Big[x^3\Big]_0^2 = \frac{1}{2}(8) = 4$. **(A)**.

Bài tập luyện tập

Let $f(x) = 6 - x^2$ on $[-1, 2]$. Find the average value of f on this interval, and find every value of c guaranteed by the Mean Value Theorem for Integrals.

Lời giải chi tiết

$f_{\text{avg}} = \frac{1}{3}\int_{-1}^2 (6 - x^2)\, dx = \frac{1}{3}\Big[6x - \frac{x^3}{3}\Big]_{-1}^2 = \frac{1}{3}\Big[(12 - \frac{8}{3}) - (-6 + \frac{1}{3})\Big] = \frac{1}{3}\Big[\frac{28}{3} + \frac{17}{3}\Big] = \frac{1}{3}\cdot 15 = 5$.
Solve $6 - c^2 = 5 \Rightarrow c^2 = 1 \Rightarrow c = \pm 1$; both lie in $[-1, 2]$, so $c = 1$ **or** $c = -1$ both satisfy the theorem.

Bài tập luyện tập

A cyclist's speed over a 4-hour ride is modeled by $v(t) = -t^2 + 4t + 3$ (mph) for $0 \le t \le 4$. Find the average speed over the ride.

Lời giải chi tiết

$v_{\text{avg}} = \frac{1}{4}\int_0^4 (-t^2 + 4t + 3)\, dt = \frac{1}{4}\Big[-\frac{t^3}{3} + 2t^2 + 3t\Big]_0^4 = \frac{1}{4}\Big[-\frac{64}{3} + 32 + 12\Big] = \frac{1}{4}\Big[44 - \frac{64}{3}\Big] = \frac{1}{4}\cdot\frac{68}{3} = \frac{68}{12} = \frac{17}{3}$ mph.

Bài tập luyện tập

A particle moves with velocity $v(t) = t^2 - 2t - 3$ on $[0, 4]$. The total distance traveled is:

(A) $\frac{34}{3}$
(C) $-\frac{20}{3}$
(B) $\frac{20}{3}$
(D) 4

Lời giải chi tiết

$v(t) = (t-3)(t+1)$; on $[0,4]$ the only relevant zero is $t = 3$. Test: $v(0) = -3 < 0$, $v(4) = 5 > 0$, so $v < 0$ on $[0,3)$ and $v > 0$ on $(3,4]$. With antiderivative $P(t) = \frac{t^3}{3} - t^2 - 3t$: $P(0) = 0$, $P(3) = 9-9-9 = -9$, $P(4) = \frac{64}{3} - 16 - 12 = -\frac{20}{3}$. Distance $= |P(3)-P(0)| + |P(4)-P(3)| = |-9| + \left|-\frac{20}{3} + 9\right| = 9 + \frac{7}{3} = \frac{27}{3} + \frac{7}{3} = \frac{34}{3}$. **(A).** (Net displacement is $P(4) - P(0) = -\frac{20}{3}$ — do not confuse it with total distance.)

Bài tập luyện tập

A particle has acceleration $a(t) = 2t - 6$ (m/s^2), initial velocity $v(0) = 5$ m/s, and initial position $s(0) = 0$. (a) Find $v(t)$ and $s(t)$. (b) Find the particle's displacement and total distance traveled on $[0,6]$.

Lời giải chi tiết

$v(t) = 5 + \int_0^t (2\tau - 6)\,d\tau = 5 + (t^2 - 6t) = t^2 - 6t + 5 = (t-1)(t-5)$. $s(t) = \int_0^t v(\tau)\,d\tau = \frac{t^3}{3} - 3t^2 + 5t$.

Zeros of v: $t = 1, 5$; on $[0,6]$, $v > 0$ on $[0,1)$, $v < 0$ on $(1,5)$, $v > 0$ on $(5,6]$. Values: $s(0) = 0$, $s(1) = \frac{1}{3} - 3 + 5 = \frac{7}{3}$, $s(5) = \frac{125}{3} - 75 + 25 = -\frac{25}{3}$, $s(6) = 72 - 108 + 30 = -6$.

Displacement $= s(6) - s(0) = -6$ m. **Total distance** $= |s(1)-s(0)| + |s(5)-s(1)| + |s(6)-s(5)| = \frac{7}{3} + \left|-\frac{25}{3} - \frac{7}{3}\right| + \left|-6 + \frac{25}{3}\right| = \frac{7}{3} + \frac{32}{3} + \frac{7}{3} = \frac{46}{3}$ m.

Bài tập luyện tập

The velocity of a particle moving along a line is negative and increasing on the interval $(2,5)$. Which must be true on $(2,5)$?

(A) Position increasing, speed increasing
(C) Position increasing, speed decreasing
(B) Position decreasing, speed decreasing
(D) Position decreasing, speed increasing

Lời giải chi tiết

$v < 0$ means position is decreasing. Speed is $|v|$; since v is negative and increasing (moving toward 0 from below), $|v|$ is decreasing. **(B).**

Bài tập luyện tập

Given $f'(x) = 3x^2 - 2$ and $f(1) = 5$, find $f(4)$.

Lời giải chi tiết

By the Net Change Theorem, $f(4) = f(1) + \int_1^4 (3x^2 - 2)\,dx = 5 + \Big[x^3 - 2x\Big]_1^4 = 5 + [(64 - 8) - (1 - 2)] = 5 + (56 + 1) = 5 + 57 = 62$.

Bài tập luyện tập

During a storm, the rate of water flowing into a retention pond, $R(t)$ (thousand liters/hour), is: $R(0) = 12$, $R(3) = 9$, $R(5) = 4$, $R(6) = 1$. Use a trapezoidal sum over these subintervals to estimate the total inflow over $[0, 6]$ hours.

Lời giải chi tiết

$$\int_0^6 R(t)\,dt \approx (3-0)\tfrac{12+9}{2} + (5-3)\tfrac{9+4}{2} + (6-5)\tfrac{4+1}{2}$$
$$= 3(10.5) + 2(6.5) + 1(2.5) = 31.5 + 13 + 2.5 = 47 \text{ thousand liters.}$$

Bài tập luyện tập

A city water tower has inflow rate $R_{\text{in}}(t) = t + 7$ and outflow rate $R_{\text{out}}(t) = 15$ (thousand gallons/min), for $0 \le t \le 10$. At $t = 0$ the tower holds 50 thousand gallons. Find the time at which the tower's volume is at a *minimum*, and find that minimum volume.

Lời giải chi tiết

Net rate: $R_{\text{net}}(t) = (t + 7) - 15 = t - 8$, negative (draining) for $t < 8$ and positive (filling) for $t > 8$, so volume is minimized at $t = 8$.

$$Q(8) = 50 + \int_0^8 (t-8)\,dt = 50 + \Big[\tfrac{t^2}{2} - 8t\Big]_0^8 = 50 + (32 - 64) = 50 - 32 = 18 \text{ thousand gallons.}$$

Bài tập luyện tập

A tank's inflow and outflow rates satisfy $\int_2^5 [R_{\text{in}}(t) - R_{\text{out}}(t)]\,dt = -120$ (liters). Which must be true?

(A) The tank gained 120 liters on $[2, 5]$

(B) The tank lost 120 liters on $[2, 5]$

(C) Inflow exceeded outflow for the entire interval

(D) The tank's volume was constant on $[2, 5]$

Lời giải chi tiết

The integral of (rate in − rate out) equals the net change in volume; a value of -120 means the volume decreased by 120 liters over $[2, 5]$. **(B)**.

Bài tập luyện tập

Find the area enclosed by $y = 8 - x^2$ and $y = x^2$.

(A) $\dfrac{64}{3}$ **(C)** $\dfrac{16}{3}$

(B) $\dfrac{32}{3}$ **(D)** 8

Lời giải chi tiết

Intersections: $8 - x^2 = x^2 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$. Top curve is $y = 8 - x^2$ (test $x = 0$: $8 > 0$).

$$A = \int_{-2}^{2} \left[(8 - x^2) - x^2\right] dx = \int_{-2}^{2} (8 - 2x^2)\, dx = \left[8x - \tfrac{2x^3}{3}\right]_{-2}^{2}$$
$$= \left(16 - \tfrac{16}{3}\right) - \left(-16 + \tfrac{16}{3}\right) = 32 - \tfrac{32}{3} = \tfrac{64}{3}.$$

(A).

Bài tập luyện tập

Find the total area enclosed between $y = x^3$ and $y = 4x$.

Lời giải chi tiết

Intersections: $x^3 = 4x \Rightarrow x(x^2 - 4) = 0 \Rightarrow x = -2, 0, 2$. Test $x = -1$: $x^3 - 4x = -1 + 4 = 3 > 0$ so $x^3 > 4x$ on $(-2, 0)$; test $x = 1$: $1 - 4 = -3 < 0$ so $4x > x^3$ on $(0, 2)$. Using $H(x) = \frac{x^4}{4} - 2x^2$: $H(-2) = 4 - 8 = -4$, $H(0) = 0$, $H(2) = -4$.

$$A = \left|H(0) - H(-2)\right| + \left|H(2) - H(0)\right| = |0 - (-4)| + |-4 - 0| = 4 + 4 = 8.$$

Bài tập luyện tập

Find the area of the region bounded by $y = \sqrt{x}$, $y = 0$, and $x = 9$.

(A) 18 **(C)** 9

(B) 27 **(D)** 6

Lời giải chi tiết

$A = \displaystyle\int_0^9 \sqrt{x}\, dx = \left[\tfrac{2}{3}x^{3/2}\right]_0^9 = \tfrac{2}{3}(27) = 18$. **(A).**

Bài tập luyện tập

Find the area enclosed by $x = 4 - y^2$ and $x = 0$.

Lời giải chi tiết

Intersections: $4-y^2=0 \Rightarrow y=\pm 2$. On $(-2,2)$, $x=4-y^2 \geq 0$, so it lies to the right of $x=0$.

$$A=\int_{-2}^{2}\left[(4-y^2)-0\right]dy=\left[4y-\tfrac{y^3}{3}\right]_{-2}^{2}=\left(8-\tfrac{8}{3}\right)-\left(-8+\tfrac{8}{3}\right)=16-\tfrac{16}{3}=\tfrac{32}{3}.$$

Bài tập luyện tập

Find the area enclosed by $x=y^2-2$ and $x=y$.

(A) $\dfrac{9}{2}$

(B) 4

(C) 5

(D) $\dfrac{7}{2}$

Lời giải chi tiết

Intersections: $y^2-2=y \Rightarrow y^2-y-2=0 \Rightarrow (y-2)(y+1)=0 \Rightarrow y=-1,2$. Test $y=0$: $x=y=0$ versus $x=y^2-2=-2$, so $x=y$ is to the right.

$$A=\int_{-1}^{2}\left[y-(y^2-2)\right]dy=\int_{-1}^{2}(y-y^2+2)\,dy=\left[\tfrac{y^2}{2}-\tfrac{y^3}{3}+2y\right]_{-1}^{2}.$$

At $y=2$: $2-\frac{8}{3}+4=6-\frac{8}{3}=\frac{10}{3}$. At $y=-1$: $\frac{1}{2}+\frac{1}{3}-2=-\frac{7}{6}$. $A=\frac{10}{3}+\frac{7}{6}=\frac{20}{6}+\frac{7}{6}=\frac{27}{6}=\frac{9}{2}$. **(A)**.

Bài tập luyện tập

Find the total area enclosed between $y=x^3-9x$ and the x-axis, for $-3 \leq x \leq 3$.

Lời giải chi tiết

Zeros: $x^3-9x=x(x^2-9)=0 \Rightarrow x=-3,0,3$. Antiderivative $H(x)=\frac{x^4}{4}-\frac{9x^2}{2}$: $H(-3)=\frac{81}{4}-\frac{81}{2}=-\frac{81}{4}$, $H(0)=0$, $H(3)=-\frac{81}{4}$.

$$A=\left|H(0)-H(-3)\right|+\left|H(3)-H(0)\right|=\left|\tfrac{81}{4}\right|+\left|-\tfrac{81}{4}\right|=\tfrac{81}{4}+\tfrac{81}{4}=\tfrac{81}{2}.$$

Bài tập luyện tập

The base of a solid is the region between $y=x$ and $y=x^2$ on $[0,1]$. Cross-sections perpendicular to the x-axis are squares with side equal to the distance between the curves. Find the volume.

Lời giải chi tiết

Side $s(x)=x-x^2$ (since $x \geq x^2$ on $[0,1]$). $A(x)=(x-x^2)^2=x^2-2x^3+x^4$.

$$V=\int_{0}^{1}(x^2-2x^3+x^4)\,dx=\left[\tfrac{x^3}{3}-\tfrac{x^4}{2}+\tfrac{x^5}{5}\right]_{0}^{1}=\tfrac{1}{3}-\tfrac{1}{2}+\tfrac{1}{5}=\tfrac{10}{30}-\tfrac{15}{30}+\tfrac{6}{30}=\tfrac{1}{30}.$$

Bài tập luyện tập

The base of a solid is the triangular region bounded by $y = 3 - x$ and the axes, for $0 \le x \le 3$. Cross-sections perpendicular to the x-axis are squares with side $y = 3 - x$. The volume is:

(A) 9
(C) 3
(B) 27
(D) 18

Lời giải chi tiết

$V = \int_0^3 (3-x)^2\,dx$. Substituting $u = 3 - x$: $\int_0^3 u^2\,du = \left[\frac{u^3}{3}\right]_0^3 = 9$. **(A)**.

Bài tập luyện tập

The base of a solid is the region under $y = \sqrt{x}$ on $[0, 4]$. Cross-sections perpendicular to the x-axis are rectangles whose height is twice the base, where the base equals $\sqrt{x}$. Find the volume.

Lời giải chi tiết

Base $= \sqrt{x}$, height $= 2\sqrt{x}$, so $A(x) = \sqrt{x} \cdot 2\sqrt{x} = 2x$.

$$V = \int_0^4 2x\,dx = \left[x^2\right]_0^4 = 16.$$

Bài tập luyện tập

The base of a solid is bounded by $y = 2 - \frac{x}{2}$ and the x-axis, for $0 \le x \le 4$. Cross-sections perpendicular to the x-axis are equilateral triangles with side $2 - \frac{x}{2}$. Find the volume.

Lời giải chi tiết

$A(x) = \frac{\sqrt{3}}{4}\left(2 - \frac{x}{2}\right)^2$. Let $w = 2 - \frac{x}{2}$, so $dx = -2\,dw$; as $x : 0 \to 4$, $w : 2 \to 0$.

$$V = \frac{\sqrt{3}}{4}\int_0^4 \left(2 - \tfrac{x}{2}\right)^2 dx = \frac{\sqrt{3}}{4} \cdot 2 \int_0^2 w^2\,dw = \frac{\sqrt{3}}{2}\left[\tfrac{w^3}{3}\right]_0^2 = \frac{\sqrt{3}}{2} \cdot \tfrac{8}{3} = \frac{4\sqrt{3}}{3}.$$

Bài tập luyện tập

The base of a solid is bounded by $y = x$ and $y = -x$, for $0 \le x \le 2$. Cross-sections perpendicular to the x-axis are semicircles with diameter equal to the distance between the two lines, $2x$. The volume is:

(A) $\frac{4\pi}{3}$
(C) $\frac{2\pi}{3}$
(B) $\frac{8\pi}{3}$
(D) 4π

Lời giải chi tiết

Radius $= x$, so $A(x) = \frac{1}{2}\pi x^2$.

$$V = \int_0^2 \tfrac{\pi}{2}x^2\,dx = \tfrac{\pi}{2}\left[\tfrac{x^3}{3}\right]_0^2 = \tfrac{\pi}{2}\cdot\tfrac{8}{3} = \tfrac{4\pi}{3}.$$

(A).

Bài tập luyện tập

The region under $y = x^3$ on $[0, 1]$ is revolved about the x-axis. The volume is:

(A) $\frac{\pi}{7}$ **(C)** $\frac{\pi}{6}$

(B) $\frac{\pi}{4}$ **(D)** $\frac{\pi}{3}$

Lời giải chi tiết

$V = \pi\int_0^1 (x^3)^2\,dx = \pi\int_0^1 x^6\,dx = \pi\left[\tfrac{x^7}{7}\right]_0^1 = \tfrac{\pi}{7}$. **(A).** (Squaring x^3 gives x^6, not x^3 again — a common exponent slip.)

Bài tập luyện tập

The region bounded by $x = \sqrt{y}$, the y-axis, and $y = 4$ is revolved about the y-axis. Find the volume.

Lời giải chi tiết

$R(y) = \sqrt{y}$, so $[R(y)]^2 = y$.

$$V = \pi\int_0^4 y\,dy = \pi\left[\tfrac{y^2}{2}\right]_0^4 = \pi(8) = 8\pi.$$

Bài tập luyện tập

The region bounded above by $y = x^2$, below by the line $y = -3$, and on the right by $x = 2$ (for $0 \le x \le 2$) is revolved about the line $y = -3$. Find the volume.

Lời giải chi tiết

The region's lower boundary is the axis itself, so this is a disk: $R(x) = x^2 - (-3) = x^2 + 3$.

$$[R(x)]^2 = x^4 + 6x^2 + 9, \qquad V = \pi\int_0^2 (x^4 + 6x^2 + 9)\,dx = \pi\left[\tfrac{x^5}{5} + 2x^3 + 9x\right]_0^2.$$

At $x = 2$: $\frac{32}{5} + 16 + 18 = \frac{32}{5} + 34 = \frac{202}{5}$. $V = \dfrac{202\pi}{5}$.

Bài tập luyện tập

The region bounded on the right by $x = \sqrt{y}$, on the left by the line $x = -2$, for $0 \le y \le 4$, is revolved about the line $x = -2$. Find the volume.

Lời giải chi tiết

The region's left boundary is the axis itself, so this is a disk: $R(y) = \sqrt{y} - (-2) = \sqrt{y} + 2$.

$$[R(y)]^2 = y + 4\sqrt{y} + 4, \qquad V = \pi \int_0^4 \left(y + 4\sqrt{y} + 4\right) dy = \pi \left[\tfrac{y^2}{2} + \tfrac{8}{3}y^{3/2} + 4y\right]_0^4.$$

At $y = 4$: $8 + \frac{8}{3}(8) + 16 = 8 + \frac{64}{3} + 16 = 24 + \frac{64}{3} = \frac{136}{3}$. $V = \dfrac{136\pi}{3}$.

Bài tập luyện tập

The region between $y = x$ and $y = x^2$ on $[0, 1]$ is revolved about the x-axis. The volume is:

(A) $\dfrac{2\pi}{15}$

(B) $\dfrac{\pi}{15}$

(C) $\dfrac{2\pi}{5}$

(D) $\dfrac{\pi}{5}$

Lời giải chi tiết

Outer radius $R(x) = x$ (since $x \ge x^2$ on $[0, 1]$), inner radius $r(x) = x^2$.

$$V = \pi \int_0^1 (x^2 - x^4)\, dx = \pi \left[\tfrac{x^3}{3} - \tfrac{x^5}{5}\right]_0^1 = \pi\left(\tfrac{1}{3} - \tfrac{1}{5}\right) = \pi \cdot \tfrac{2}{15} = \tfrac{2\pi}{15}.$$

(A).

Bài tập luyện tập

The region bounded by $y = 4 - x^2$ and $y = 0$ is revolved about the line $y = -2$. Find the volume.

Lời giải chi tiết

Curve meets axis at $x = \pm 2$. Outer radius (to the farther curve $y = 4 - x^2$): $R(x) = (4 - x^2) - (-2) = 6 - x^2$. Inner radius (to the nearer boundary $y = 0$): $r(x) = 0 - (-2) = 2$.

$$[R(x)]^2 - [r(x)]^2 = (6 - x^2)^2 - 4 = 36 - 12x^2 + x^4 - 4 = 32 - 12x^2 + x^4.$$

$$V = \pi \int_{-2}^{2} (32 - 12x^2 + x^4)\, dx = \pi \left[32x - 4x^3 + \tfrac{x^5}{5}\right]_{-2}^{2}.$$

At $x = 2$: $64 - 32 + \frac{32}{5} = 32 + \frac{32}{5} = \frac{192}{5}$; by symmetry double this: $V = \pi \cdot \frac{384}{5} = \dfrac{384\pi}{5}$.

Bài tập luyện tập

The region bounded by $x = y^2$ and $x = 2y$ (for $0 \le y \le 2$) is revolved about the y-axis. Find the volume.

Lời giải chi tiết

Since $2y \ge y^2$ on $[0, 2]$, outer radius $R(y) = 2y$, inner radius $r(y) = y^2$.

$$V = \pi \int_0^2 [(2y)^2 - (y^2)^2]\, dy = \pi \int_0^2 (4y^2 - y^4)\, dy = \pi \left[\tfrac{4y^3}{3} - \tfrac{y^5}{5}\right]_0^2.$$

At $y = 2$: $\frac{32}{3} - \frac{32}{5} = 32\left(\frac{1}{3} - \frac{1}{5}\right) = 32 \cdot \frac{2}{15} = \frac{64}{15}$. $V = \dfrac{64\pi}{15}$.

Bài tập luyện tập

Let R be the region enclosed by $y = x^2$ and $y = 2x$. (a) Find the area of R. (b) Find the volume when R is revolved about the x-axis.

Lời giải chi tiết

Intersections: $x^2 = 2x \Rightarrow x = 0, 2$. On $(0, 2)$, $2x \ge x^2$.

(a) $A = \displaystyle\int_0^2 (2x - x^2)\, dx = \left[x^2 - \tfrac{x^3}{3}\right]_0^2 = 4 - \tfrac{8}{3} = \tfrac{4}{3}$.

(b) Washer about the x-axis: outer $R(x) = 2x$, inner $r(x) = x^2$.

$$V = \pi \int_0^2 [(2x)^2 - (x^2)^2]\, dx = \pi \int_0^2 (4x^2 - x^4)\, dx = \pi \left[\tfrac{4x^3}{3} - \tfrac{x^5}{5}\right]_0^2 = \pi\left(\tfrac{32}{3} - \tfrac{32}{5}\right) = \pi \cdot \tfrac{64}{15} = \tfrac{64\pi}{15}.$$

Bài tập luyện tập

Let R be the same region enclosed by $y = x^2$ and $y = 2x$ from the previous problem. Find the volume when R is revolved about the line $y = -1$.

Lời giải chi tiết

Since the axis $y = -1$ lies below the entire region, distances increase by 1: outer radius (to the lower curve $y = x^2$, farther from $y = -1$) is $R(x) = x^2 + 1$; inner radius (to the upper curve $y = 2x$, nearer to $y = -1$) is $r(x) = 2x + 1$.

$$[R(x)]^2 - [r(x)]^2 = (x^2 + 1)^2 - (2x + 1)^2 = (x^4 + 2x^2 + 1) - (4x^2 + 4x + 1) = x^4 - 2x^2 - 4x.$$

$$V = \pi \int_0^2 (x^4 - 2x^2 - 4x)\, dx = \pi \left[\tfrac{x^5}{5} - \tfrac{2x^3}{3} - 2x^2\right]_0^2.$$

At $x = 2$: $\frac{32}{5} - \frac{16}{3} - 8$. Common denominator 15: $\frac{96}{15} - \frac{80}{15} - \frac{120}{15} = -\frac{104}{15}$. Taking the (necessarily positive) volume integral: $V = \pi \cdot \frac{104}{15} = \dfrac{104\pi}{15}$.

Bài tập luyện tập

Let R be the region bounded by $y = \sqrt{x}$, the x-axis, and $x = 4$. (a) Find the area of R. (b) Find the volume when R is revolved about the line $x = 4$.

Lời giải chi tiết

(a) $A = \displaystyle\int_0^4 \sqrt{x}\,dx = \left[\tfrac{2}{3}x^{3/2}\right]_0^4 = \tfrac{2}{3}(8) = \tfrac{16}{3}$.

(b) Revolving about the vertical line $x = 4$ requires integrating in y; rewrite the curve as $x = y^2$, with y ranging from 0 to 2 (since $y = \sqrt{4} = 2$). The region's right boundary is the axis $x = 4$ itself, so this is a disk with radius $R(y) = 4 - y^2$ (distance from the curve to the line $x = 4$).

$$[R(y)]^2 = 16 - 8y^2 + y^4, \qquad V = \pi \int_0^2 (16 - 8y^2 + y^4)\,dy = \pi\left[16y - \tfrac{8y^3}{3} + \tfrac{y^5}{5}\right]_0^2.$$

At $y = 2$: $32 - \frac{64}{3} + \frac{32}{5}$. Common denominator 15: $\frac{480}{15} - \frac{320}{15} + \frac{96}{15} = \frac{256}{15}$. $V = \dfrac{256\pi}{15}$.

Bài tập luyện tập

Let R be the region bounded by $y = x^2$ and $y = 4$. (a) Find the area of R. (b) Find the volume when R is revolved about the x-axis.

Lời giải chi tiết

Intersections: $x^2 = 4 \Rightarrow x = \pm 2$.

(a) $A = \displaystyle\int_{-2}^2 (4 - x^2)\,dx = \left[4x - \tfrac{x^3}{3}\right]_{-2}^2 = \left(8 - \tfrac{8}{3}\right) - \left(-8 + \tfrac{8}{3}\right) = 16 - \tfrac{16}{3} = \tfrac{32}{3}$.

(b) Even though the lower curve $y = x^2$ touches the x-axis at $x = 0$, the region R itself is bounded *below by* $y = x^2$, not by $y = 0$ — so revolving about the x-axis produces a **washer**, not a disk, with outer radius $R(x) = 4$ (constant, the horizontal line) and inner radius $r(x) = x^2$:

$$V = \pi \int_{-2}^2 \left[4^2 - (x^2)^2\right] dx = \pi \int_{-2}^2 (16 - x^4)\,dx = \pi\left[16x - \tfrac{x^5}{5}\right]_{-2}^2.$$

At $x = 2$: $32 - \frac{32}{5} = \frac{128}{5}$; by symmetry double this: $V = \pi \cdot \frac{256}{5} = \dfrac{256\pi}{5}$.

Bài tập luyện tập

A solid has known cross-sections perpendicular to the x-axis that are squares, with base region between $y = f(x)$ and $y = g(x)$ on $[a, b]$, where $f(x) > g(x)$ throughout. Which integral gives the volume?

(A) $\pi \displaystyle\int_a^b [f(x) - g(x)]^2\,dx$

(B) $\displaystyle\int_a^b [f(x) - g(x)]^2\,dx$

(C) $\displaystyle\int_a^b [f(x)^2 - g(x)^2]\,dx$

(D) $\pi \displaystyle\int_a^b [f(x)^2 - g(x)^2]\,dx$

Lời giải chi tiết

The cross-sections are squares, not circles, so there is **no factor of** π anywhere in the formula — π only appears for circular cross-sections (disks/washers). The side length of each square is the distance between the curves, $f(x) - g(x)$, so the cross-sectional area is $[f(x) - g(x)]^2$ and $V = \int_a^b [f(x) - g(x)]^2\, dx$. **(B).** (Choice (C) is the mistake of subtracting before squaring in a washer-style setup, which does not apply here at all since this is not a solid of revolution.)

Ôn tập nhanh — toàn bộ chương trình:

- **Giới hạn & liên tục:** luôn thử thế trực tiếp trước; nếu gặp $\frac{0}{0}$ thì phân tích nhân tử, nhân liên hợp, hoặc dùng các giới hạn đặc biệt $\lim_{x\to 0} \frac{\sin x}{x} = 1$; giới hạn hai phía tồn tại khi và chỉ khi hai giới hạn một phía bằng nhau.
- **Đạo hàm cơ bản & quy tắc chuỗi:** áp dụng quy tắc chuỗi cho *mọi* hàm hợp, kể cả khi ẩn dưới dạng đạo hàm ẩn (implicit) hay đạo hàm hàm ngược; với đạo hàm ẩn, luôn nhân thêm $\frac{dy}{dx}$ mỗi khi đạo hàm một biểu thức chứa y.
- **Ứng dụng đạo hàm theo ngữ cảnh:** tốc độ liên quan (related rates) luôn bắt đầu từ một phương trình liên hệ các đại lượng rồi lấy đạo hàm theo t; xấp xỉ tuyến tính dùng $f(a) + f'(a)(x - a)$; luôn ghi rõ đơn vị của đạo hàm trong bài toán thực tế.
- **Phân tích đồ thị hàm số:** $f' > 0 \Rightarrow f$ tăng, $f' < 0 \Rightarrow f$ giảm; $f'' > 0 \Rightarrow$ lõm lên (concave up), $f'' < 0 \Rightarrow$ lõm xuống (concave down); điểm uốn chỉ được công nhận khi f'' **thực sự đổi dấu** qua điểm đó, không chỉ đơn thuần $f'' = 0$.
- **Tích phân & Định lý cơ bản (FTC):** $\int_a^b f'(x)\, dx = f(b) - f(a)$; luôn cộng hằng số $+C$ với tích phân bất định; kiểm tra lại đạo hàm của đáp số để chắc chắn tính đúng.
- **Phương trình vi phân:** tách biến rồi lấy tích phân hai vế; mô hình tăng trưởng/phân rã mũ $\frac{dy}{dt} = ky \Rightarrow y = y_0 e^{kt}$; luôn dùng điều kiện đầu để tìm hằng số C (hoặc y_0) cụ thể.
- **Chuyển động & tích lũy:** độ dời $= \int v\, dt$ (có dấu), quãng đường $= \int |v|\, dt$ (luôn không âm) — hai đại lượng này khác nhau bất cứ khi nào v đổi dấu trên đoạn đang xét.
- **Diện tích giữa hai đường cong:** luôn tìm giao điểm trước; kiểm tra đường nào ở trên bằng cách thử một điểm ở giữa; nếu hai đường cắt nhau nhiều lần, tách tích phân tại từng giao điểm rồi lấy trị tuyệt đối từng mảnh.
- **Thể tích khối:** luôn hình dung (hoặc vẽ) một lát cắt đại diện trước khi lập tích phân; với thiết diện đã biết, xác định đúng độ dài cạnh/đường kính rồi mới áp công thức diện tích hình học tương ứng; với đĩa/vành khăn, đo bán kính là khoảng cách **thực sự** từ trục quay đến đường cong (không mặc định $R = f(x)$ nếu trục không phải là trục hoành); với vành khăn, luôn bình phương từng bán kính **trước khi** trừ.

Parametric Equations, Polar Coordinates, and Vector-Valued Functions

Mục tiêu Unit: Mô tả đường cong bằng phương trình tham số, xác định chiều chuyển động và khử tham số; tính dy/dx và d^2y/dx^2 của đường tham số, tìm tiếp tuyến ngang/đứng và xét tính lồi–lõm (concavity); tính độ dài cung của đường tham số; đạo hàm và tích phân hàm vectơ, dựng lại vị trí từ vận tốc với điều kiện đầu; giải bài toán chuyển động trên mặt phẳng (vận tốc, gia tốc, tốc độ $=$ độ dài vectơ vận tốc, quãng đường, độ dời, vị trí) cả có và không có máy tính; chuyển đổi giữa toạ độ cực và Descartes, nhận dạng các đường cực quen thuộc (đường tròn, cardioid, limaçon, hoa hồng), tính hệ số góc tiếp tuyến và hiểu ý nghĩa của $dr/d\theta$; tính diện tích miền cực $\frac{1}{2}\int r^2\,d\theta$ và diện tích giữa hai đường cực, kể cả khi hai đường giao nhau tại gốc toạ độ.

9.1 Defining and Differentiating Parametric Equations

9.1.1 Parametric equations and orientation

Many curves in the plane are not graphs of functions $y = f(x)$: a circle fails the vertical line test, and the path of a moving object may loop back over itself. A **parametric** description solves this by writing *both* coordinates as functions of a third variable, the **parameter** (usually t, often thought of as time):

$$x = f(t), \qquad y = g(t), \qquad t \text{ in some interval } I.$$

As t increases through I, the point $(f(t), g(t))$ traces out a curve C. The direction in which the point moves as t increases is the **orientation** of the curve, and it is shown on a sketch with arrows. Two different parametrizations can trace the same set of points at different speeds or in opposite directions — the parametric equations carry more information than the picture alone.

Parametric curve

A **parametric curve** is the set of points $(x(t), y(t))$ for t in an interval I, where x and y are continuous functions of t. The point at $t = a$ is the **initial point**, the point at $t = b$ the **terminal point** (for $I = [a, b]$), and the **orientation** is the direction of increasing t.

Eliminating the parameter. To identify the shape of a parametric curve, solve one equation for t and substitute into the other, or use an identity such as $\cos^2 t + \sin^2 t = 1$. The resulting Cartesian equation describes the *set* of points, but it loses the orientation and may include points the parametric curve never reaches, so always restrict x and y to the values actually attained.

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Phương trình tham số mô tả *cả* x lẫn y theo một biến thứ ba t (tham số, thường hiểu là thời gian). Nhờ vậy ta mô tả được những đường không phải đồ thị hàm số (đường tròn, đường tự cắt) và biết thêm **chiều chuyển động** (orientation) — chiều mà điểm di chuyển khi t tăng, vẽ bằng mũi tên. **Khử tham số** (rút t từ một phương trình rồi thế vào phương trình kia, hoặc dùng $\cos^2 t + \sin^2 t = 1$) cho ta phương trình Descartes để nhận dạng hình, nhưng sẽ *mất* thông tin về chiều và có thể thừa điểm — phải giới hạn lại miền giá trị x, y thực sự đạt được.

Ví dụ mẫu · Eliminating the parameter (algebraic)

Sketch the curve $x = t^2 - 1, y = 2t$ for $-2 \le t \le 2$, indicate its orientation, and find a Cartesian equation.

Eliminate t. From $y = 2t$ we get $t = \frac{y}{2}$, so $x = \left(\frac{y}{2}\right)^2 - 1$, i.e.

$$\boxed{x = \tfrac{1}{4}y^2 - 1}, \qquad -4 \le y \le 4.$$

This is a parabola opening to the right with vertex $(-1, 0)$.

Orientation. At $t = -2$ the point is $(3, -4)$; at $t = 0$ it is $(-1, 0)$; at $t = 2$ it is $(3, 4)$. Since $y = 2t$ always increases, the curve is traced *from bottom to top*: starting at $(3, -4)$, passing through the vertex, ending at $(3, 4)$. The restriction $-4 \le y \le 4$ is essential — the Cartesian equation alone describes the whole infinite parabola.

Ví dụ mẫu · Eliminating the parameter (trigonometric)

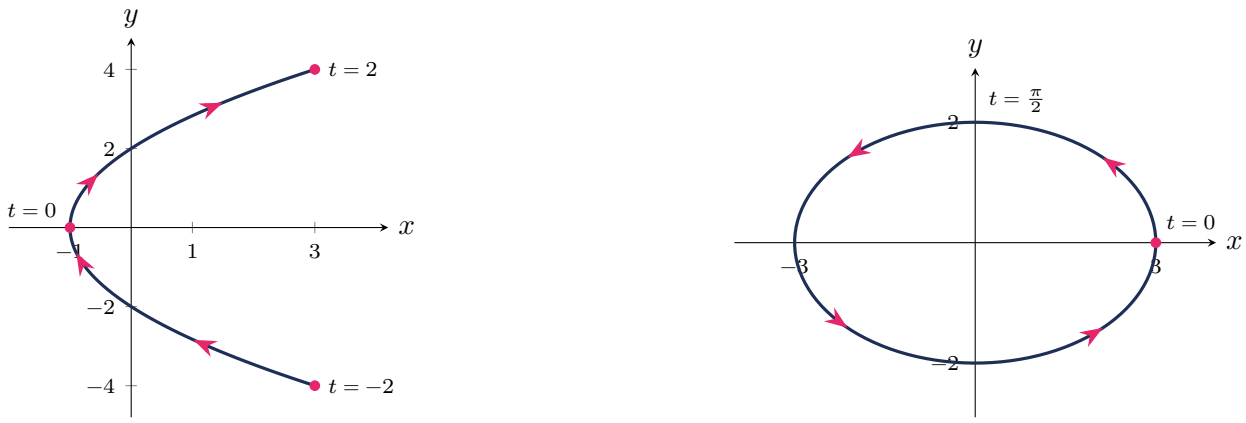
Identify the curve $x = 3\cos t, y = 2\sin t, 0 \le t \le 2\pi$, and its orientation.

Use the Pythagorean identity. $\cos t = \frac{x}{3}$ and $\sin t = \frac{y}{2}$, so $\cos^2 t + \sin^2 t = 1$ gives

$$\boxed{\frac{x^2}{9} + \frac{y^2}{4} = 1},$$

an ellipse with semi-axes 3 (horizontal) and 2 (vertical).

Orientation. $t = 0$: $(3, 0)$; $t = \frac{\pi}{2}$: $(0, 2)$; $t = \pi$: $(-3, 0)$. The point moves **counterclockwise**, and it traces the ellipse exactly once on $[0, 2\pi]$. (By contrast, $x = 3\cos t$, $y = -2\sin t$ traces the same ellipse clockwise, and $x = 3\cos 2t$, $y = 2\sin 2t$ on $[0, 2\pi]$ traces it twice.)



Left: $x = t^2 - 1,\ y = 2t$ on $[-2, 2]$ — half of the parabola $x = \frac{1}{4}y^2 - 1$ is traced upward. Right: $x = 3\cos t,\ y = 2\sin t$ traces the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ counterclockwise. Pink arrows show the orientation (direction of increasing t).

9.1.2 The derivative dy/dx of a parametric curve

If $x = x(t)$ and $y = y(t)$ are differentiable and $\frac{dx}{dt} \ne 0$, then near that value of t the curve is locally the graph of a differentiable function y of x. The chain rule gives $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$, and solving for $\frac{dy}{dx}$:

Slope of a parametric curve

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \qquad \text{provided } \frac{dx}{dt} \neq 0.$$

- **Horizontal tangent:** $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$.
- **Vertical tangent:** $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$.
- If *both* $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$ at $t = a$, the formula gives no information: the curve may have a cusp, a corner, or an ordinary tangent there. Investigate with $\lim_{t \to a} \frac{dy/dt}{dx/dt}$.

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Hệ số góc tiếp tuyến của đường tham số: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ (lấy "tốc độ theo phương đứng" chia "tốc độ theo phương ngang"). **Tiếp tuyến ngang** khi tử bằng 0, mẫu khác 0; **tiếp tuyến đứng** khi mẫu bằng 0, tử khác 0. Nếu *cả hai* cùng bằng 0 thì chưa kết luận được — phải tính giới hạn của tỉ số (có thể là điểm lùi/cusp, điểm gãy, hoặc tiếp tuyến bình thường).

Ví dụ mẫu · Tangent line at a given value of t

A curve is defined by $x = t^2 + t$, $y = t^3 - 3t$. Find the equation of the line tangent to the curve at the point where $t = 2$.

Point: $x(2) = 4 + 2 = 6$, $y(2) = 8 - 6 = 2$, so the point is $(6, 2)$.

Slope: $\frac{dx}{dt} = 2t + 1$, $\frac{dy}{dt} = 3t^2 - 3$. At $t = 2$: $\frac{dx}{dt} = 5$, $\frac{dy}{dt} = 9$, so $\left.\frac{dy}{dx}\right|_{t=2} = \frac{9}{5}$.

Tangent line: $\boxed{y - 2 = \frac{9}{5}(x - 6)}$. On the AP exam, point-slope form is acceptable; there is no need to simplify.

Ví dụ mẫu · Horizontal and vertical tangents; a curve that crosses itself

Consider the curve C given by $x = t^2$, $y = t^3 - 3t$ for all real t.
(a) Find all points on C where the tangent line is horizontal or vertical.
(b) Show that C passes through $(3, 0)$ twice, and find the slope of each tangent line there.

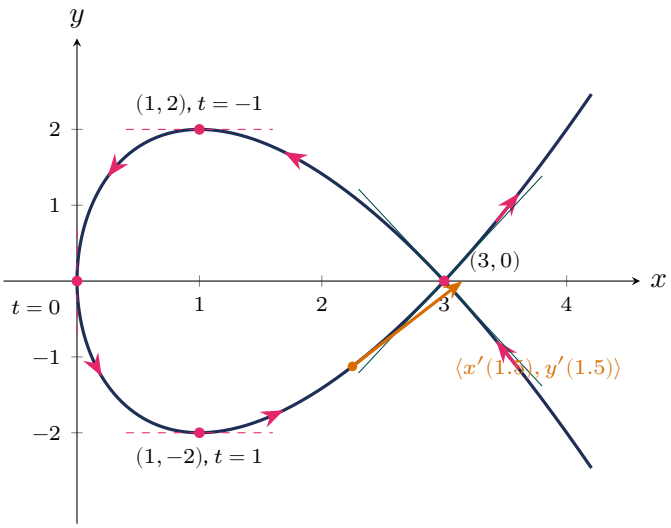
(a) $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 3t^2 - 3 = 3(t - 1)(t + 1)$.

- Horizontal: $\frac{dy}{dt} = 0 \Rightarrow t = \pm 1$, and there $\frac{dx}{dt} = \pm 2 \neq 0$. Points: $t = 1 \Rightarrow (1, -2)$; $t = -1 \Rightarrow (1, 2)$.
- Vertical: $\frac{dx}{dt} = 0 \Rightarrow t = 0$, and $\frac{dy}{dt} = -3 \neq 0$. Point: $(0, 0)$.

(b) $y = t(t^2 - 3) = 0$ at $t = 0, \pm\sqrt{3}$. For $t = \pm\sqrt{3}$, $x = 3$, so $(3, 0)$ is reached at two different times — the curve crosses itself. The slopes are

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t}: \qquad t = \sqrt{3}: \ \frac{6}{2\sqrt{3}} = \sqrt{3}, \qquad t = -\sqrt{3}: \ \frac{6}{-2\sqrt{3}} = -\sqrt{3}.$$

Two different tangent lines, $y = \sqrt{3}(x - 3)$ and $y = -\sqrt{3}(x - 3)$, pass through the crossing point.



The curve $x = t^2,\ y = t^3 - 3t$. Horizontal tangents at $(1, \pm 2)$, a vertical tangent at the origin, and two tangent lines (slopes $\pm\sqrt{3}$) at the self-intersection $(3, 0)$. The orange arrow (drawn at 30% length) is the velocity vector at $t = 1.5$; it is always tangent to the path and points in the direction of motion.

Ví dụ mẫu · When both derivatives vanish

For $x = t^3$, $y = t^2$, find $\dfrac{dy}{dx}$ and describe the curve at $t = 0$.

$\dfrac{dx}{dt} = 3t^2$, $\dfrac{dy}{dt} = 2t$, so for $t \neq 0$, $\dfrac{dy}{dx} = \dfrac{2t}{3t^2} = \dfrac{2}{3t}$. At $t = 0$ both derivatives are 0, so the formula is undefined. Examine the limit: as $t \to 0^+$, $\frac{2}{3t} \to +\infty$; as $t \to 0^-$, $\frac{2}{3t} \to -\infty$. Eliminating t gives $y = x^{2/3}$: the curve has a **cusp** at the origin with a vertical tangent. Neither "horizontal tangent" nor "vertical tangent by the rule" may be concluded just from $0/0$.

(!) Lỗi thường gặp: Viết $\dfrac{dy}{dx} = \dfrac{dx/dt}{dy/dt}$ (đảo ngược) là lỗi rất hay gặp — nhớ "y trên, x dưới" giống như ký hiệu $\frac{dy}{dx}$. Lỗi thứ hai: kết luận "tiếp tuyến ngang" chỉ vì $\frac{dy}{dt} = 0$ mà quên kiểm tra $\frac{dx}{dt} \neq 0$. Lỗi thứ ba: khi khử tham số, quên giới hạn miền giá trị (ví dụ $x = t^2$ nên $x \geq 0$).

9.2 Second Derivatives of Parametric Equations

The second derivative $\frac{d^2y}{dx^2}$ is the derivative *with respect to* x of the first derivative $\frac{dy}{dx}$. Since $\frac{dy}{dx}$ is expressed as a function of t, we apply the same "divide by $\frac{dx}{dt}$" rule a second time:

Second derivative of a parametric curve

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}, \qquad \frac{dx}{dt} \neq 0.$$

The curve is **concave up** where $\frac{d^2y}{dx^2} > 0$ and **concave down** where $\frac{d^2y}{dx^2} < 0$.

Warning. In general $\dfrac{d^2y}{dx^2} \neq \dfrac{d^2y/dt^2}{d^2x/dt^2}$. Always differentiate the expression for $\frac{dy}{dx}$ (as a function of t) with respect to t, then divide by $\frac{dx}{dt}$.

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Đạo hàm cấp hai: lấy đạo hàm của $\frac{dy}{dx}$ (đang là biểu thức theo t) *theo* t, rồi **chia thêm một lần nữa cho** $\frac{dx}{dt}$. Tuyệt đối **không** lấy $\frac{y''(t)}{x''(t)}$ — đó là lỗi sai kinh điển. Dấu của $\frac{d^2y}{dx^2}$ cho biết đường cong lõm lên (concave up, >0) hay lõm xuống (concave down, <0) khi nhìn theo phương x.

Ví dụ mẫu · Concavity of $x = t^2,\ y = t^3 - 3t$

Find $\dfrac{d^2y}{dx^2}$ for the curve $x = t^2$, $y = t^3 - 3t$ and determine where the curve is concave up.

First derivative: $\dfrac{dy}{dx} = \dfrac{3t^2 - 3}{2t} = \dfrac{3}{2}t - \dfrac{3}{2t}$.

Differentiate with respect to t: $\dfrac{d}{dt}\left(\dfrac{3}{2}t - \dfrac{3}{2}t^{-1}\right) = \dfrac{3}{2} + \dfrac{3}{2t^2} = \dfrac{3t^2+3}{2t^2}$.

Divide by $dx/dt = 2t$:

$$\frac{d^2y}{dx^2} = \frac{(3t^2+3)/(2t^2)}{2t} = \boxed{\frac{3(t^2+1)}{4t^3}}.$$

The numerator is always positive, so the sign is the sign of t^3: **concave up for** $t > 0$ (the branch from the origin down through $(1,-2)$ and up through $(3,0)$) and concave down for $t < 0$. Compare with the figure in Section 9.1: the branch for $t > 0$ bends upward, the branch for $t < 0$ bends downward.

Ví dụ mẫu · The cycloid

A wheel of radius 1 rolls along the x-axis; a point on its rim traces the **cycloid** $x = t - \sin t$, $y = 1 - \cos t$. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for $0 < t < 2\pi$, and describe the concavity.

$\dfrac{dx}{dt} = 1 - \cos t$, $\dfrac{dy}{dt} = \sin t$, so $\dfrac{dy}{dx} = \dfrac{\sin t}{1 - \cos t}$.

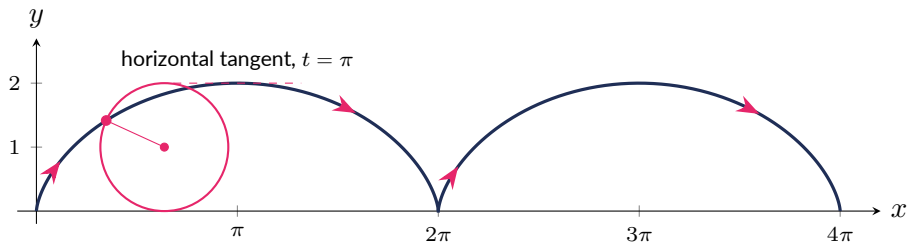
Quotient rule in t:

$$\frac{d}{dt}\left(\frac{\sin t}{1-\cos t}\right) = \frac{\cos t(1-\cos t) - \sin t \cdot \sin t}{(1-\cos t)^2} = \frac{\cos t - (\cos^2 t + \sin^2 t)}{(1-\cos t)^2}$$
$$= \frac{\cos t - 1}{(1-\cos t)^2} = \frac{-1}{1-\cos t}.$$

Divide by $\frac{dx}{dt} = 1 - \cos t$:

$$\frac{d^2y}{dx^2} = \boxed{\frac{-1}{(1-\cos t)^2}} < 0 \quad \text{for } 0 < t < 2\pi.$$

Each arch of the cycloid is **concave down** everywhere. At $t = \pi$ (top of the arch), $\frac{dy}{dx} = \frac{0}{2} = 0$: horizontal tangent. At $t = 0$ and $t = 2\pi$ both derivatives vanish; $\frac{dy}{dx} \to \pm\infty$, producing cusps where the point touches the ground.



Two arches of the cycloid $x = t - \sin t,\ y = 1 - \cos t$ traced by a point on a rolling unit circle. Each arch is concave down ($\frac{d^2y}{dx^2} < 0$) with cusps at $x = 0, 2\pi, 4\pi$.

Ví dụ mẫu · Second derivative at a point

For $x = e^t$, $y = t^2 e^t$, find $\dfrac{d^2y}{dx^2}$ at $t = 1$.

$\dfrac{dx}{dt} = e^t$, $\dfrac{dy}{dt} = (2t + t^2)e^t$, so $\dfrac{dy}{dx} = \dfrac{(2t + t^2)e^t}{e^t} = t^2 + 2t$.

$\dfrac{d}{dt}(t^2 + 2t) = 2t + 2$, and $\dfrac{d^2y}{dx^2} = \dfrac{2t + 2}{e^t}$. At $t = 1$: $\dfrac{d^2y}{dx^2} = \boxed{\dfrac{4}{e}} \approx 1.472 > 0$, so the curve is concave up there.

Check of the warning: $\frac{y''(t)}{x''(t)} = \frac{(t^2+4t+2)e^t}{e^t} = t^2 + 4t + 2 = 7$ at $t = 1$ — a completely different (wrong) number.

9.3 Finding Arc Lengths of Curves Given by Parametric Equations

Over a short time interval Δt, the point moves horizontally by $\Delta x \approx x'(t)\Delta t$ and vertically by $\Delta y \approx y'(t)\Delta t$. By the Pythagorean theorem the small piece of curve has length

$$\Delta s \approx \sqrt{(\Delta x)^2 + (\Delta y)^2} \approx \sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2}\ \Delta t.$$

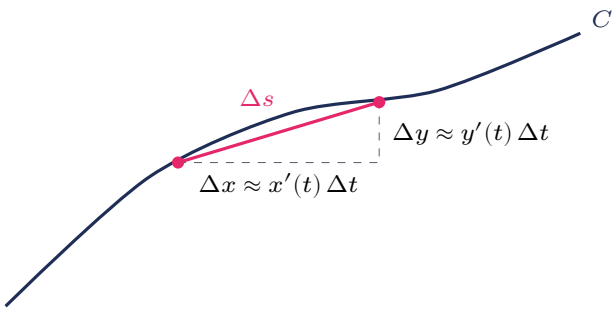
Summing these pieces and letting $\Delta t \to 0$ gives a definite integral.

Arc length of a parametric curve

If $x'(t)$ and $y'(t)$ are continuous on $[a, b]$ and the curve is traced **exactly once** as t goes from a to b, its length is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt.$$

For a function graph $y = f(x)$, $a \le x \le b$ (take $x = t$), this reduces to the Unit 8 formula $L = \int_a^b \sqrt{1 + \big(f'(x)\big)^2}\ dx$.



Each small piece of the curve is approximately the hypotenuse of a right triangle with legs Δx and Δy; adding up $\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$ leads to the arc length integral.

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Độ dài cung của đường tham số: $L = \int_a^b \sqrt{(x')^2 + (y')^2}\,dt$ — chính là định lý Pytago áp dụng cho từng đoạn nhỏ rồi cộng lại. Điều kiện quan trọng: đường cong phải được vẽ **đúng một lần** khi t chạy từ a đến b; nếu điểm đi lại trên một đoạn đã đi thì tích phân cho *quãng đường đi được* (lớn hơn độ dài hình học của đường). Biểu thức dưới dấu căn chính là bình phương **tốc độ** — nên độ dài cung = tích phân của tốc độ (xem mục 9.6).

Ví dụ mẫu · Exact arc length

Find the length of the curve $x = t^2$, $y = \frac{2}{3}t^3$ for $0 \le t \le \sqrt{3}$.

$\frac{dx}{dt} = 2t$, $\frac{dy}{dt} = 2t^2$, so $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4t^2 + 4t^4 = 4t^2(1+t^2)$. For $t \ge 0$, $\sqrt{4t^2(1+t^2)} = 2t\sqrt{1+t^2}$. Substitute $u = 1 + t^2$, $du = 2t\,dt$ (so $t = 0 \mapsto u = 1$, $t = \sqrt{3} \mapsto u = 4$):

$$L = \int_0^{\sqrt{3}} 2t\sqrt{1+t^2}\,dt = \int_1^4 u^{1/2}\,du = \frac{2}{3}\Big[u^{3/2}\Big]_1^4 = \frac{2}{3}(8-1) = \boxed{\frac{14}{3}}.$$

Ví dụ mẫu · Length of one arch of the cycloid

Find the length of one arch of the cycloid $x = t - \sin t$, $y = 1 - \cos t$, $0 \le t \le 2\pi$.

$(x')^2 + (y')^2 = (1 - \cos t)^2 + \sin^2 t = 1 - 2\cos t + \cos^2 t + \sin^2 t = 2 - 2\cos t$. Using the half-angle identity $1 - \cos t = 2\sin^2 \frac{t}{2}$: $\sqrt{2 - 2\cos t} = \sqrt{4\sin^2 \frac{t}{2}} = 2\left|\sin \frac{t}{2}\right| = 2\sin \frac{t}{2}$ on $[0, 2\pi]$ (where $\sin \frac{t}{2} \ge 0$).

$$L = \int_0^{2\pi} 2\sin\frac{t}{2}\,dt = \Big[-4\cos\frac{t}{2}\Big]_0^{2\pi} = -4(-1) - (-4)(1) = \boxed{8}.$$

A remarkable fact: one arch has length exactly 8 — four times the diameter of the rolling wheel.

Ví dụ mẫu · Arc length with a calculator

(Calculator allowed.) Find the length of the loop of the curve $x = t^2$, $y = t^3 - 3t$ (see the figure in Section 9.1).

Limits. The loop starts and ends at the self-intersection $(3, 0)$, reached at $t = -\sqrt{3}$ and $t = \sqrt{3}$. Between these values the loop is traced exactly once.

Integral. $(x')^2 + (y')^2 = (2t)^2 + (3t^2 - 3)^2$, so

$$L = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{4t^2 + (3t^2 - 3)^2}\,dt \approx \boxed{10.735}.$$

On the exam, write the integral with its limits, then give the calculator value to three decimal places.

Ví dụ mẫu · Traced once versus traced twice

The curve $x = \cos 2t$, $y = \sin 2t$, $0 \le t \le 2\pi$, is the unit circle. Compute $\int_0^{2\pi} \sqrt{(x')^2 + (y')^2}\,dt$ and interpret.

$x' = -2\sin 2t$, $y' = 2\cos 2t$, so $\sqrt{(x')^2 + (y')^2} = \sqrt{4(\sin^2 2t + \cos^2 2t)} = 2$, and $\int_0^{2\pi} 2\,dt = 4\pi$. But the circumference is 2π: the circle is traced **twice** on $[0, 2\pi]$. The integral measures distance traveled, not the length of the set of points. The circle's length is $\int_0^{\pi} 2\,dt = 2\pi$.

(!) Lỗi thường gặp: Quên dấu căn, hoặc viết $\sqrt{(x')^2 + (y')^2} = x' + y'$ (sai!). Khi rút căn của một bình phương phải lấy trị tuyệt đối, ví dụ $\sqrt{4\sin^2 \frac{t}{2}} = 2\left|\sin \frac{t}{2}\right|$, rồi xét dấu trên khoảng tích phân. Với câu có máy tính, chỉ cần viết đúng tích phân (kèm cận) rồi bấm máy — không cần tìm nguyên hàm.

9.4 Defining and Differentiating Vector-Valued Functions

9.4.1 Vector-valued functions

A **vector-valued function** assigns a vector to each value of t:

$$\vec{r}(t) = \langle x(t),\, y(t)\rangle \qquad (\text{also written } x(t)\,\mathbf{i} + y(t)\,\mathbf{j}).$$

When we draw $\vec{r}(t)$ as an arrow from the origin, its tip is the point $(x(t), y(t))$. As t varies the tip traces exactly the parametric curve $x = x(t)$, $y = y(t)$. So a vector-valued function is a parametric curve viewed as a moving **position vector**; the vector language makes velocity and acceleration easy to describe.

9.4.2 Derivatives: velocity and acceleration

Vector functions are differentiated **component by component**:

Derivative of a vector-valued function

If $\vec{r}(t) = \langle x(t), y(t)\rangle$ is the position of a particle, then

$$\vec{v}(t) = \vec{r}\,'(t) = \langle x'(t),\, y'(t)\rangle \quad \text{(velocity vector)},$$

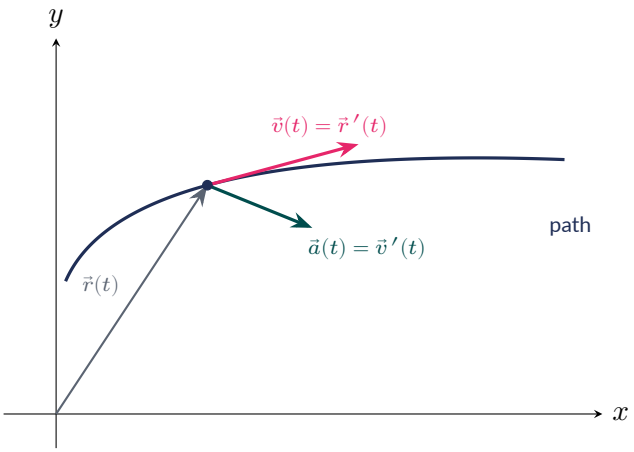
$$\vec{a}(t) = \vec{v}\,'(t) = \langle x''(t),\, y''(t)\rangle \quad \text{(acceleration vector)}.$$

The velocity vector at time t is **tangent to the path** and points in the direction of motion; its slope is $\frac{y'(t)}{x'(t)} = \frac{dy}{dx}$. The **speed** is the magnitude of velocity:

$$\text{speed} = \|\vec{v}(t)\| = \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}.$$

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Hàm vectơ $\vec{r}(t) = \langle x(t), y(t)\rangle$ là "mũi tên vị trí" từ gốc toạ độ đến điểm đang chuyển động; đầu mũi tên vẽ nên đúng đường tham số. Đạo hàm **từng thành phần**: $\vec{v} = \langle x', y'\rangle$ là **vectơ vận tốc** (luôn tiếp xúc với quỹ đạo, chỉ theo chiều chuyển động), $\vec{a} = \langle x'', y''\rangle$ là **vectơ gia tốc**. **Tốc độ** (speed) là *độ dài* của vectơ vận tốc $\sqrt{(x')^2 + (y')^2}$ — một số không âm, không phải vectơ.



The position vector $\vec{r}(t)$ points from the origin to the particle. The velocity vector is tangent to the path in the direction of motion; the acceleration vector points toward the side to which the path is bending (here $\vec{r}(t) = \langle t^2,\ 1 + 2t - \frac{1}{2}t^2 \rangle$ at $t = 1.2$; vectors drawn shortened).

Ví dụ mẫu · Velocity, acceleration, and speed

A particle moves in the plane with position $\vec{r}(t) = \langle e^{2t},\ \sin(\pi t) \rangle$. Find the velocity vector, the acceleration vector, and the speed of the particle at $t = 0$ and at $t = 1$.

$\vec{v}(t) = \langle 2e^{2t},\ \pi\cos(\pi t) \rangle, \quad \vec{a}(t) = \langle 4e^{2t},\ -\pi^2 \sin(\pi t) \rangle.$

At $t = 0$: $\vec{v}(0) = \langle 2, \pi \rangle$, $\vec{a}(0) = \langle 4, 0 \rangle$, speed $= \sqrt{4 + \pi^2} \approx 3.724$.

At $t = 1$: $\vec{v}(1) = \langle 2e^2, -\pi \rangle$, $\vec{a}(1) = \langle 4e^2, 0 \rangle$, speed $= \sqrt{4e^4 + \pi^2} \approx 15.108$. At $t = 1$ the particle is moving right ($x' > 0$) and down ($y' < 0$).

Ví dụ mẫu · Direction of motion from the velocity vector

A particle has position $\vec{r}(t) = \langle t^3 - 3t^2,\ t^2 - 4t \rangle$ for $t \geq 0$. (a) For what values of t is the particle moving to the left? (b) Is the particle ever at rest? (c) Find the slope of the path at $t = 3$.

$\vec{v}(t) = \langle 3t^2 - 6t,\ 2t - 4 \rangle = \langle 3t(t-2),\ 2(t-2) \rangle.$

(a) Moving left means $x'(t) < 0$: $3t(t-2) < 0$ for $\boxed{0 < t < 2}$.

(b) At rest means $\vec{v} = \vec{0}$, i.e. *both* components are 0. $x'(t) = 0$ at $t = 0, 2$; $y'(t) = 0$ at $t = 2$. Both vanish at $\boxed{t = 2}$: the particle is momentarily at rest at $\vec{r}(2) = \langle -4, -4 \rangle$.

(c) $\dfrac{dy}{dx}\Big|_{t=3} = \dfrac{y'(3)}{x'(3)} = \dfrac{2}{9}.$

(!) Lỗi thường gặp: **Tốc độ** (speed) là một *số*, **vận tốc** (velocity) là một *vectơ*. Đề hỏi "velocity vector" thì trả lời $\langle\ ,\ \rangle$; hỏi "speed" thì trả lời một số. "Đứng yên" (at rest) đòi hỏi *cả hai* thành phần vận tốc bằng 0 cùng lúc — chỉ $x' = 0$ thì hạt vẫn có thể đang chuyển động thẳng đứng.

9.5 Integrating Vector-Valued Functions

Integration of vector functions is also done component by component. Indefinite integrals produce a **vector constant** $\vec{C} = \langle C_1, C_2 \rangle$, which is determined by an initial condition.

Integrals of vector-valued functions

$$\int \langle f(t), g(t)\rangle\, dt = \left\langle \int f(t)\, dt,\ \int g(t)\, dt \right\rangle + \vec{C},$$

$$\int_a^b \langle f(t), g(t)\rangle\, dt = \left\langle \int_a^b f(t)\, dt,\ \int_a^b g(t)\, dt \right\rangle.$$

Position from velocity (Fundamental Theorem, applied to each coordinate):

$$\vec{r}(t) = \vec{r}(t_0) + \int_{t_0}^{t} \vec{v}(s)\, ds, \qquad \text{i.e.} \qquad x(t) = x(t_0) + \int_{t_0}^{t} x'(s)\, ds, \quad y(t) = y(t_0) + \int_{t_0}^{t} y'(s)\, ds.$$

Likewise $\vec{v}(t) = \vec{v}(t_0) + \int_{t_0}^{t} \vec{a}(s)\, ds$.

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Tích phân hàm vectơ cũng làm **từng thành phần**. Tích phân bất định cho ra một **hằng vectơ** $\vec{C} = \langle C_1, C_2\rangle$, xác định nhờ điều kiện đầu. Công thức then chốt cho bài toán chuyển động: **vị trí lúc** t = **vị trí ban đầu** + **tích phân vận tốc** (áp dụng riêng cho x và cho y). Đây chính là Định lý Biến thiên Ròng của Unit 8, chỉ là làm hai lần.

Ví dụ mẫu · Position from velocity and an initial condition

A particle has velocity $\vec{v}(t) = \langle 2t,\ \cos(\pi t)\rangle$ and position $\vec{r}(0) = \langle 1, 3\rangle$. Find $\vec{r}(t)$ and the position at $t = 2$.

Integrate each component: $\vec{r}(t) = \left\langle t^2 + C_1,\ \dfrac{\sin(\pi t)}{\pi} + C_2 \right\rangle$.

At $t = 0$: $\langle C_1,\ C_2\rangle = \langle 1, 3\rangle$. Hence $\boxed{\vec{r}(t) = \left\langle t^2 + 1,\ 3 + \dfrac{\sin(\pi t)}{\pi} \right\rangle}$ and $\vec{r}(2) = \langle 5, 3\rangle$.

Ví dụ mẫu · From acceleration to position

A projectile-like particle has acceleration $\vec{a}(t) = \langle 6t,\ -2\rangle$, initial velocity $\vec{v}(0) = \langle 0, 4\rangle$ and initial position $\vec{r}(0) = \langle 1, 1\rangle$. Find $\vec{r}(t)$, and find the maximum y-coordinate of the particle for $t \geq 0$.

Velocity: $\vec{v}(t) = \langle 3t^2 + C_1,\ -2t + C_2\rangle$; $\vec{v}(0) = \langle 0, 4\rangle$ gives $\vec{v}(t) = \langle 3t^2,\ 4 - 2t\rangle$.

Position: $\vec{r}(t) = \langle t^3 + D_1,\ 4t - t^2 + D_2\rangle$; $\vec{r}(0) = \langle 1, 1\rangle$ gives $\boxed{\vec{r}(t) = \langle t^3 + 1,\ 1 + 4t - t^2\rangle}$.

Highest point: $y'(t) = 4 - 2t$ changes from positive to negative at $t = 2$, so y has its absolute maximum for $t \geq 0$ at $t = 2$: $y(2) = 1 + 8 - 4 = \boxed{5}$ (at the point $(9, 5)$).

Ví dụ mẫu · A definite integral of a vector function

Evaluate $\displaystyle\int_0^1 \left\langle \frac{1}{1+t^2},\ te^{t^2} \right\rangle dt$.

First component: $\displaystyle\int_0^1 \frac{dt}{1+t^2} = \Big[\arctan t\Big]_0^1 = \frac{\pi}{4}$. Second component (with $u = t^2$):

$$\int_0^1 te^{t^2}\,dt = \left[\tfrac{1}{2}e^{t^2}\right]_0^1 = \frac{e-1}{2}.$$

$$\int_0^1 \left\langle \frac{1}{1+t^2},\ te^{t^2}\right\rangle dt = \boxed{\left\langle \frac{\pi}{4},\ \frac{e-1}{2}\right\rangle}.$$

If this were a velocity, the answer would be the **displacement vector** of the particle from $t=0$ to $t=1$.

9.6 Solving Motion Problems Using Parametric and Vector-Valued Functions

Planar motion toolkit

For a particle with position $(x(t), y(t))$ and velocity $\vec{v}(t) = \langle x'(t), y'(t)\rangle$ on $a \le t \le b$:

Quantity	How to compute
Velocity / acceleration vector	$\langle x'(t), y'(t)\rangle$ / $\langle x''(t), y''(t)\rangle$
Speed	$\sqrt{(x'(t))^2 + (y'(t))^2}$
Slope of path	$\dfrac{dy}{dx} = \dfrac{y'(t)}{x'(t)}$
Moving right / left	$x'(t) > 0$ / $x'(t) < 0$ (up / down: $y'(t) > 0$ / $y'(t) < 0$)
Position at time b	$x(b) = x(a) + \int_a^b x'(t)\,dt$, $y(b) = y(a) + \int_a^b y'(t)\,dt$
Displacement vector	$\left\langle \int_a^b x'(t)\,dt,\ \int_a^b y'(t)\,dt\right\rangle$
Total distance traveled	$\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt = \int_a^b \text{speed}\,dt$
Speed increasing?	speed increases when $\vec{v}\cdot\vec{a} > 0$; check $\frac{d}{dt}$(speed), or compare speeds at two times

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Bộ công cụ chuyển động trên mặt phẳng: **tốc độ** $= \sqrt{(x')^2 + (y')^2}$; **quãng đường đi được** $= \int_a^b$ tốc độ dt (luôn dương, cộng dồn mọi đoạn); **độ dời** là *vectơ* $\langle \int x'\,dt, \int y'\,dt\rangle$ (chỉ phụ thuộc điểm đầu và điểm cuối); **vị trí** $=$ vị trí ban đầu $+$ tích phân vận tốc, làm riêng cho x và y. Sang trái/phải xét dấu x'; lên/xuống xét dấu y'. Câu có máy tính: viết đúng tích phân rồi bấm máy, làm tròn **3 chữ số thập phân**, ghi đơn vị nếu đề cho.

Ví dụ mẫu · Circular motion: distance versus displacement

A particle moves with $\vec{r}(t) = \langle 3\cos 2t,\ 3\sin 2t\rangle$ for $0 \le t \le \pi$. Find its speed, the total distance traveled, and the displacement vector.

$\vec{v}(t) = \langle -6\sin 2t,\ 6\cos 2t\rangle$, so speed $= \sqrt{36\sin^2 2t + 36\cos^2 2t} = 6$ (constant).

Total distance $= \displaystyle\int_0^{\pi} 6\,dt = \boxed{6\pi}$ (one full lap of a circle of radius 3).

Displacement $= \vec{r}(\pi) - \vec{r}(0) = \langle 3, 0\rangle - \langle 3, 0\rangle = \boxed{\langle 0, 0\rangle}$: the particle returns to its starting point.

Note also $\vec{v}\cdot\vec{a} = 0$ here (constant speed), even though $\vec{a} = \langle -12\cos 2t, -12\sin 2t\rangle \neq \vec{0}$.

Ví dụ mẫu · Non-calculator motion analysis

A particle moves with $x(t) = t^3 - 6t^2$ and $y(t) = 2t^2 - 8t$ for $0 \le t \le 5$.
(a) Find the speed of the particle at $t = 2$.
(b) At what times is the particle moving vertically (straight up or down)?
(c) Find the displacement vector from $t = 0$ to $t = 4$.
(d) At $t = 1$, is the speed of the particle increasing or decreasing? Justify.

$\vec{v}(t) = \langle 3t^2 - 12t,\ 4t - 8\rangle, \quad \vec{a}(t) = \langle 6t - 12,\ 4\rangle.$

(a) $\vec{v}(2) = \langle 12 - 24,\ 0\rangle = \langle -12, 0\rangle$; speed $= \boxed{12}$.

(b) Moving vertically: $x'(t) = 3t(t-4) = 0$ with $y'(t) \neq 0$. At $t = 0$: $y' = -8 \neq 0$; at $t = 4$: $y' = 8 \neq 0$. So $\boxed{t = 0 \text{ and } t = 4}$.

(c) $\langle x(4) - x(0),\ y(4) - y(0)\rangle = \langle (64 - 96) - 0,\ (32 - 32) - 0\rangle = \boxed{\langle -32,\ 0\rangle}$.

(d) $\vec{v}(1) = \langle -9, -4\rangle$ and $\vec{a}(1) = \langle -6, 4\rangle$. Then $\vec{v} \cdot \vec{a} = 54 - 16 = 38 > 0$. Equivalently, $\frac{d}{dt}(\text{speed}) = \frac{x'x'' + y'y''}{\text{speed}} = \frac{38}{\sqrt{97}} > 0$. The speed is **increasing** at $t = 1$.

Why $\vec{v} \cdot \vec{a}$ decides. Speed is $\sqrt{(x')^2 + (y')^2}$; by the chain rule

$$\frac{d}{dt}(\text{speed}) = \frac{2x'x'' + 2y'y''}{2\sqrt{(x')^2 + (y')^2}} = \frac{x'x'' + y'y''}{\text{speed}},$$

whose sign is the sign of $x'x'' + y'y'' = \vec{v} \cdot \vec{a}$. (The dot product is not required on the AP exam; computing $\frac{d}{dt}(\text{speed})$ directly, or comparing speeds, earns the same credit.)

Ví dụ mẫu · AP-style free response (calculator allowed)

A particle moves in the xy-plane so that for $0 \le t \le 3$,

$$\frac{dx}{dt} = 3\sin\left(\frac{t^2}{2}\right), \qquad \frac{dy}{dt} = 1 + 2\cos t.$$

At $t = 0$ the particle is at $(2, -1)$.
(a) Find the speed of the particle at $t = 1.5$, and find its acceleration vector at $t = 1.5$.
(b) Find the total distance traveled by the particle over $0 \le t \le 3$.
(c) Find the position of the particle at $t = 3$.
(d) For $0 \le t \le 3$, find the time at which the particle is farthest to the right, and find its x-coordinate then. Justify your answer.

(a) $x'(1.5) = 3\sin(1.125) = 2.706\ldots$, $y'(1.5) = 1 + 2\cos(1.5) = 1.141\ldots$
Speed $= \sqrt{(x'(1.5))^2 + (y'(1.5))^2} = \boxed{2.938}$.
Acceleration: $x''(t) = 3t\cos\left(\frac{t^2}{2}\right)$, $y''(t) = -2\sin t$, so $\vec{a}(1.5) = \boxed{\langle 1.940,\ -1.995\rangle}$. (A calculator's numerical derivative gives the same values.)

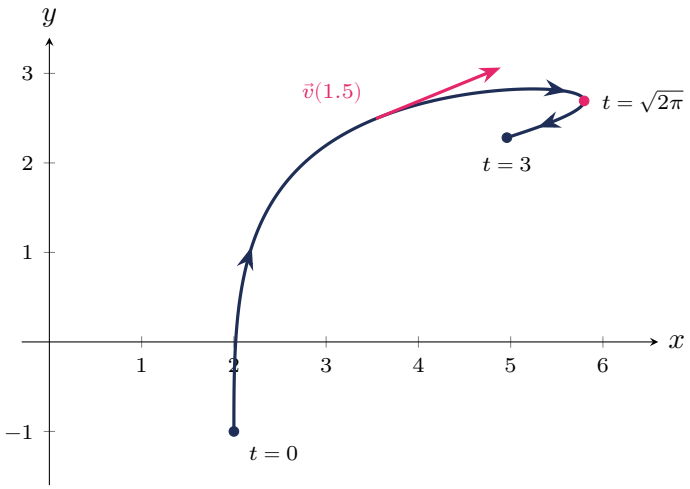
(b) Distance $= \displaystyle\int_0^3 \sqrt{\left(3\sin\tfrac{t^2}{2}\right)^2 + (1 + 2\cos t)^2}\, dt = \boxed{7.410}$.

(c) $x(3) = 2 + \displaystyle\int_0^3 3\sin\left(\tfrac{t^2}{2}\right) dt = \boxed{4.959}$, $\quad y(3) = -1 + \displaystyle\int_0^3 (1 + 2\cos t)\, dt = -1 + 3 + 2\sin 3 = \boxed{2.282}$.
Position at $t = 3$: $(4.959,\ 2.282)$.

(d) $x'(t) = 0$ when $\frac{t^2}{2} = \pi$, i.e. $t = \sqrt{2\pi} \approx 2.507$ (the only zero in $(0, 3)$, since $\frac{t^2}{2} \le 4.5 < 2\pi$). $x'(t) > 0$ on $(0, \sqrt{2\pi})$ and $x'(t) < 0$ on $(\sqrt{2\pi}, 3)$, so x increases then decreases: the absolute

maximum of x on $[0,3]$ occurs at $\boxed{t=\sqrt{2\pi}\approx 2.507}$. There

$$x(\sqrt{2\pi}) = 2 + \int_0^{\sqrt{2\pi}} 3\sin\left(\tfrac{t^2}{2}\right)\,dt = \boxed{5.796}.$$



Path of the particle in the free-response example for $0 \le t \le 3$ (computed numerically). The particle starts at $(2,-1)$, is farthest right at $t=\sqrt{2\pi}$ (pink point, $x\approx 5.796$), then turns back to the left, ending at $(4.959,\ 2.282)$. The pink arrow is the velocity vector at $t=1.5$ (drawn at half length).

Ví dụ mẫu · Tangent line and motion at a point (calculator)

For the particle in the previous example, find the slope of the path at $t=2$, and determine whether the particle is moving up or down and left or right at $t=2.8$.

$$\left.\frac{dy}{dx}\right|_{t=2} = \frac{y'(2)}{x'(2)} = \frac{1+2\cos 2}{3\sin 2} = \frac{0.16771}{2.72789} = \boxed{0.061}.$$

At $t=2.8$: $x'(2.8) = 3\sin(3.92) = -2.106 < 0$ and $y'(2.8) = 1+2\cos(2.8) = -0.884 < 0$. The particle is moving **left and down**.

(!) Lỗi thường gặp: Ba lỗi hay gặp trong FRQ chuyển động: (1) tính quãng đường bằng $\int \sqrt{x^2+y^2}\,dt$ (dùng vị trí thay vì vận tốc) — phải là $\int \sqrt{(x')^2+(y')^2}\,dt$; (2) tìm vị trí mà **quên cộng toạ độ ban đầu**: $x(3) = x(0) + \int_0^3 x'(t)\,dt$, không phải chỉ $\int_0^3 x'(t)\,dt$; (3) chỉ ghi giá trị máy tính mà không viết tích phân — rubric AP yêu cầu thấy được tích phân (cận và hàm dưới dấu tích phân).

9.7 Defining Polar Coordinates and Differentiating in Polar Form

9.7.1 Polar coordinates

In the **polar coordinate system** a point P is located by its directed distance r from a fixed point O (the **pole**, i.e. the origin) and the angle θ measured counterclockwise from the positive x-axis (the **polar axis**). We write $P=(r,\theta)$.

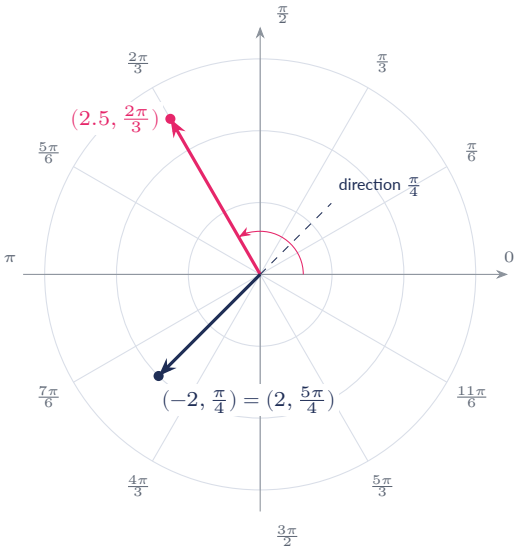
- Polar coordinates are **not unique**: (r,θ), $(r,\theta+2\pi)$, and $(-r,\theta+\pi)$ all name the same point.
- A **negative** r means: face in direction θ, then walk $|r|$ units *backward* (in direction $\theta+\pi$).

- The pole is $(0, \theta)$ for every θ.

Conversion formulas

$$x = r\cos\theta, \qquad y = r\sin\theta; \qquad r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}\ (x \neq 0).$$

When converting $(x, y) \to (r, \theta)$, choose θ in the correct quadrant: $\tan\theta = \frac{y}{x}$ alone does not determine it.



Plotting polar points. $(2.5, \frac{2\pi}{3})$: rotate $\frac{2\pi}{3}$, walk 2.5 forward. $(-2, \frac{\pi}{4})$: face direction $\frac{\pi}{4}$ (dashed), then walk 2 units backward — the same point as $(2, \frac{5\pi}{4})$.

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Toạ độ cực (r, θ): r là khoảng cách có hướng từ gốc (cực), θ là góc quay ngược chiều kim đồng hồ từ tia Ox. Một điểm có *vô số* cặp toạ độ cực; $r < 0$ nghĩa là quay mặt theo hướng θ rồi **đi lùi** $|r|$ đơn vị. Công thức chuyển đổi: $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$, $\tan\theta = \frac{y}{x}$ — nhớ chọn θ đúng góc phần tư.

Ví dụ mẫu · Converting points

(a) Convert $(r, \theta) = \left(4, \frac{2\pi}{3}\right)$ to rectangular coordinates. (b) Give two polar representations of the rectangular point $(-2, -2)$.

(a) $x = 4\cos\frac{2\pi}{3} = 4\left(-\frac{1}{2}\right) = -2$, $y = 4\sin\frac{2\pi}{3} = 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}$. Point: $\boxed{(-2,\ 2\sqrt{3})}$.

(b) $r = \sqrt{4+4} = 2\sqrt{2}$. $\tan\theta = \frac{-2}{-2} = 1$, but the point is in Quadrant III, so $\theta = \frac{5\pi}{4}$ (not $\frac{\pi}{4}$). Representations: $\boxed{\left(2\sqrt{2}, \frac{5\pi}{4}\right)}$ and, using a negative radius, $\boxed{\left(-2\sqrt{2}, \frac{\pi}{4}\right)}$.

Ví dụ mẫu · Converting equations

(a) Write $r = 6\sin\theta$ in rectangular form. (b) Write $x^2 + y^2 = 4x$ in polar form. (c) Identify $r = 2\sec\theta$.

(a) Multiply by r: $r^2 = 6r\sin\theta \Rightarrow x^2 + y^2 = 6y \Rightarrow x^2 + (y-3)^2 = 9$: a **circle** of radius 3 centered at $(0, 3)$, passing through the pole.

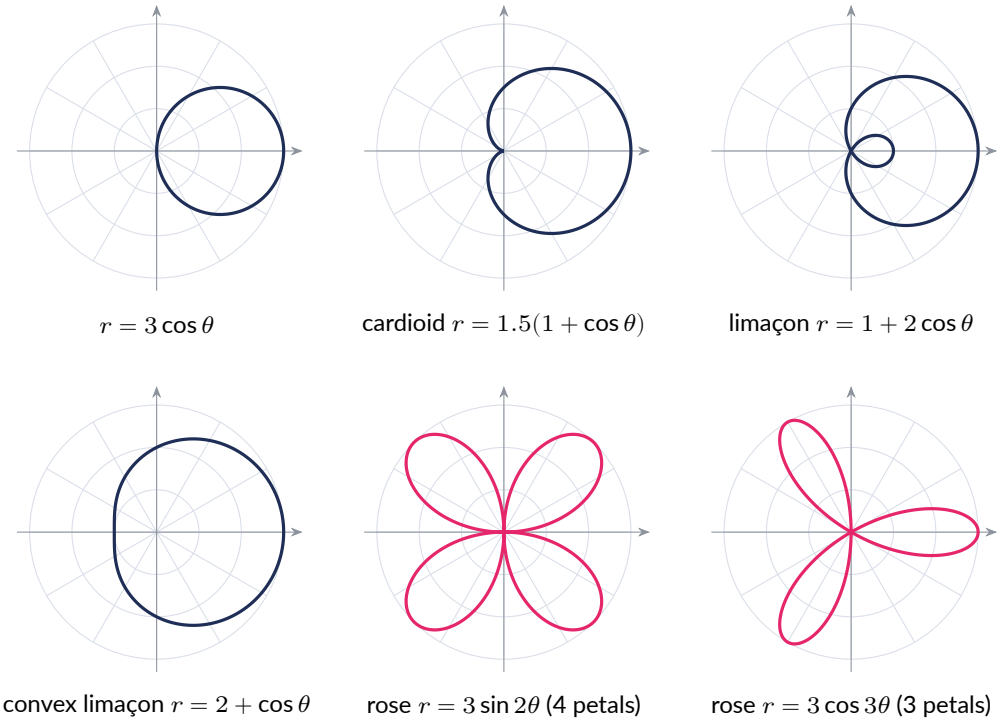
(b) $r^2 = 4r\cos\theta \Rightarrow r = 4\cos\theta$ (the case $r = 0$, the pole, is already on this curve at $\theta = \frac{\pi}{2}$).

(c) $r = \dfrac{2}{\cos\theta} \Rightarrow r\cos\theta = 2 \Rightarrow x = 2$: a **vertical line**.

9.7.2 Graphs of polar equations

A polar equation $r = f(\theta)$ is graphed by plotting $(f(\theta), \theta)$ as θ increases. The families below appear constantly on the AP exam; knowing their shapes and *over what θ-interval they are traced once* is essential for area problems.

Equation	Name / shape	Traced once on
$r = a$	circle of radius a centered at the pole	$[0, 2\pi]$
$r = 2a\cos\theta,\ r = 2a\sin\theta$	circle of radius $\lvert a\rvert$ through the pole	$[0, \pi]$
$r = a(1 \pm \cos\theta),\ r = a(1 \pm \sin\theta)$	**cardioid** (heart shape, cusp at pole)	$[0, 2\pi]$
$r = a + b\cos\theta,\ \lvert a\rvert < \lvert b\rvert$	**limaçon with an inner loop**	$[0, 2\pi]$
$r = a + b\cos\theta,\ \lvert b\rvert < \lvert a\rvert < 2\lvert b\rvert$	dimpled limaçon	$[0, 2\pi]$
$r = a + b\cos\theta,\ \lvert a\rvert \geq 2\lvert b\rvert$	convex limaçon	$[0, 2\pi]$
$r = a\cos(n\theta)$, $r = a\sin(n\theta)$, n odd	**rose** with n petals	$[0, \pi]$
$r = a\cos(n\theta)$, $r = a\sin(n\theta)$, n even	**rose** with $2n$ petals	$[0, 2\pi]$



$r = 3\cos\theta$ | cardioid $r = 1.5(1 + \cos\theta)$ | limaçon $r = 1 + 2\cos\theta$

convex limaçon $r = 2 + \cos\theta$ | rose $r = 3\sin 2\theta$ (4 petals) | rose $r = 3\cos 3\theta$ (3 petals)

Six common polar curves, each drawn on a polar grid with circles $r = 1, 2, 3$ and rays every 30°. The limaçon $r = 1 + 2\cos\theta$ has an inner loop because $r < 0$ for $\frac{2\pi}{3} < \theta < \frac{4\pi}{3}$.

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Các đường cực cần thuộc lòng: **đường tròn** ($r = a$; $r = 2a\cos\theta$ hoặc $2a\sin\theta$ đi qua gốc), **cardioid** (hình trái tim, $r = a(1 \pm \cos\theta)$), **limaçon** $r = a + b\cos\theta$ (có *vòng trong* khi $|a| < |b|$ vì r đổi dấu), **hoa hồng** $r = a\cos n\theta$ (n lẻ: n cánh, vẽ xong trên $[0, \pi]$; n chẵn: $2n$ cánh, cần $[0, 2\pi]$). Biết đường được vẽ *đúng một lần* trên khoảng nào là chìa khoá để tính diện tích không bị đếm hai lần.

9.7.3 Slopes of polar curves

A polar curve $r = f(\theta)$ is a parametric curve with parameter θ:

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta.$$

Applying the product rule and the parametric slope formula:

Slope of a polar curve

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}, \qquad \frac{dx}{d\theta} \neq 0.$$

Horizontal tangent: $\frac{dy}{d\theta} = 0$, $\frac{dx}{d\theta} \neq 0$. Vertical tangent: $\frac{dx}{d\theta} = 0$, $\frac{dy}{d\theta} \neq 0$. If $r = 0$ and $\frac{dr}{d\theta} \neq 0$ at $\theta = \alpha$, the curve passes through the pole with tangent line $\theta = \alpha$.

Ví dụ mẫu · Tangent line to a polar curve

Find the equation of the tangent line to $r = 1 + 2\cos\theta$ at $\theta = \frac{\pi}{2}$.

Point: $r\left(\frac{\pi}{2}\right) = 1$, so $(x, y) = \left(1 \cdot \cos\frac{\pi}{2},\ 1 \cdot \sin\frac{\pi}{2}\right) = (0, 1)$.

Derivatives: $\frac{dr}{d\theta} = -2\sin\theta = -2$ at $\theta = \frac{\pi}{2}$.

$$\frac{dy}{d\theta} = r'\sin\theta + r\cos\theta = (-2)(1) + (1)(0) = -2,$$

$$\frac{dx}{d\theta} = r'\cos\theta - r\sin\theta = (-2)(0) - (1)(1) = -1.$$

Slope $= \dfrac{-2}{-1} = 2$. Tangent line: $\boxed{y = 2x + 1}$.

Ví dụ mẫu · Horizontal and vertical tangents of a cardioid

Find the points on the cardioid $r = 1 + \cos\theta$, $0 \le \theta < 2\pi$, where the tangent line is horizontal or vertical.

$y = (1 + \cos\theta)\sin\theta = \sin\theta + \frac{1}{2}\sin 2\theta \ \Rightarrow\ \dfrac{dy}{d\theta} = \cos\theta + \cos 2\theta = 2\cos^2\theta + \cos\theta - 1 = (2\cos\theta - 1)(\cos\theta + 1)$.

$x = (1 + \cos\theta)\cos\theta = \cos\theta + \cos^2\theta \ \Rightarrow\ \dfrac{dx}{d\theta} = -\sin\theta - 2\sin\theta\cos\theta = -\sin\theta(1 + 2\cos\theta)$.

Horizontal ($\frac{dy}{d\theta} = 0$): $\cos\theta = \frac{1}{2}$ gives $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$, where $\frac{dx}{d\theta} = \mp\frac{\sqrt{3}}{2} \cdot 2 \neq 0$. Here $r = \frac{3}{2}$, so the points are $\left(\frac{3}{4},\ \pm\frac{3\sqrt{3}}{4}\right)$. ($\cos\theta = -1$, i.e. $\theta = \pi$, is handled below.)

Vertical ($\frac{dx}{d\theta} = 0$): $\sin\theta = 0$ gives $\theta = 0$ or π; $\cos\theta = -\frac{1}{2}$ gives $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$. At $\theta = 0$: $\frac{dy}{d\theta} = 2 \neq 0$, vertical tangent at $(2, 0)$. At $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$: $\frac{dy}{d\theta} = -\frac{1}{2} - \frac{1}{2} = -1 \neq 0$, $r = \frac{1}{2}$, vertical tangents at $\left(-\frac{1}{4},\ \pm\frac{\sqrt{3}}{4}\right)$.

At $\theta = \pi$ both derivatives are 0 ($r = 0$, the pole): this is the **cusp**. Neither rule applies; one can show $\frac{dy}{dx} \to 0$ as $\theta \to \pi$, so the cusp points along the x-axis.

9.7.4 The meaning of $dr/d\theta$

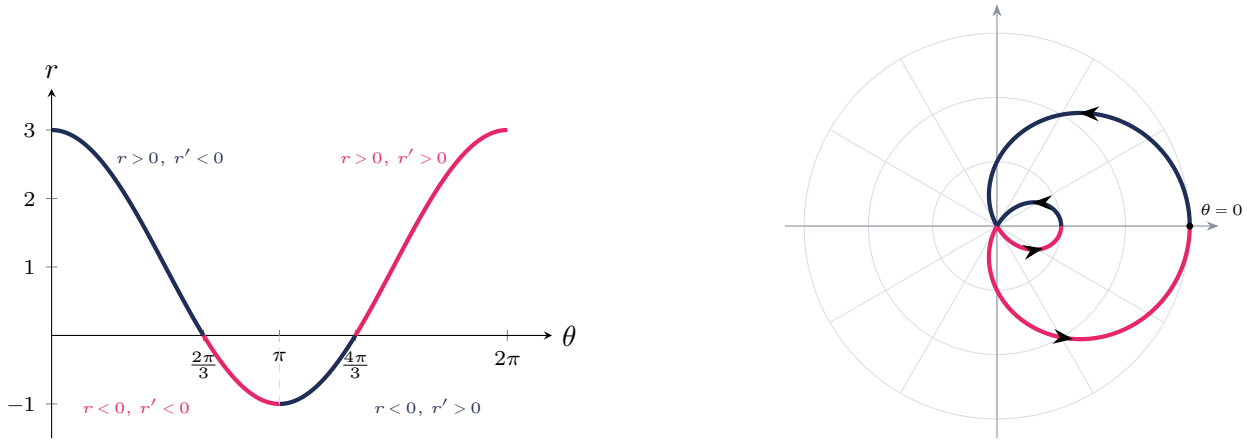
$\frac{dr}{d\theta}$ is **not** the slope of the curve. It measures how fast the directed distance r changes as the angle increases. The actual *distance* from the pole is $|r|$, so:

Moving toward or away from the pole

As θ increases, the point on $r = f(\theta)$ is

- **moving away from the pole** if r and $\frac{dr}{d\theta}$ have the **same sign** ($r > 0,\ r' > 0$ or $r < 0,\ r' < 0$);
- **moving toward the pole** if r and $\frac{dr}{d\theta}$ have **opposite signs**.

In particular, when $r > 0$: $\frac{dr}{d\theta} > 0$ means the distance from the origin is increasing.



$r = 1 + 2\cos\theta$. Left: r against θ. Right: the polar curve. Pink: r and $\frac{dr}{d\theta}$ have the same sign, so the point moves *away* from the pole. Navy: opposite signs, moving *toward* the pole. On $(\frac{2\pi}{3}, \pi)$, r is decreasing yet the point moves away — because r is negative.

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$\frac{dr}{d\theta}$ **không phải** hệ số góc tiếp tuyến! Nó là tốc độ thay đổi của r theo góc. Khoảng cách thật tới gốc là $|r|$, nên: r và $\frac{dr}{d\theta}$ **cùng dấu** ⇒ điểm đang **ra xa** gốc; **trái dấu** ⇒ đang **lại gần** gốc. Bẫy AP hay ra: khi $r < 0$ mà r đang giảm ($r' < 0$) thì $|r|$ lại *tăng* — điểm đi ra xa gốc.

Ví dụ mẫu · Interpreting $dr/d\theta$

For $r = 1 + 2\cos\theta$, find $\frac{dr}{d\theta}$ at $\theta = \frac{\pi}{3}$ and at $\theta = \frac{5\pi}{6}$, and interpret each value in terms of the distance from the origin.

$\frac{dr}{d\theta} = -2\sin\theta$.

At $\theta = \frac{\pi}{3}$: $r = 1 + 2\cdot\frac{1}{2} = 2 > 0$ and $\frac{dr}{d\theta} = -\sqrt{3} < 0$. Opposite signs: as θ increases, the point is **getting closer to the origin** (at a rate of $\sqrt{3}$ units per radian).

At $\theta = \frac{5\pi}{6}$: $r = 1 + 2\left(-\frac{\sqrt{3}}{2}\right) = 1 - \sqrt{3} < 0$ and $\frac{dr}{d\theta} = -2\cdot\frac{1}{2} = -1 < 0$. Same sign: $|r|$ is increasing, so the point is **moving away from the origin**.

Ví dụ mẫu · Rates along a polar curve (AP style)

A particle moves along the curve $r = 2 + \sin\theta$ so that $\frac{d\theta}{dt} = 3$ for all t. Find $\frac{dr}{dt}$ and $\frac{dy}{dt}$ at the instant when $\theta = \frac{\pi}{6}$.

Chain rule: $\dfrac{dr}{dt} = \dfrac{dr}{d\theta}\cdot\dfrac{d\theta}{dt} = \cos\theta\cdot 3 = 3\cdot\frac{\sqrt{3}}{2} = \boxed{\dfrac{3\sqrt{3}}{2}}$.

$y = r\sin\theta = (2 + \sin\theta)\sin\theta = 2\sin\theta + \sin^2\theta$, so $\dfrac{dy}{d\theta} = 2\cos\theta + 2\sin\theta\cos\theta$. At $\theta = \frac{\pi}{6}$:

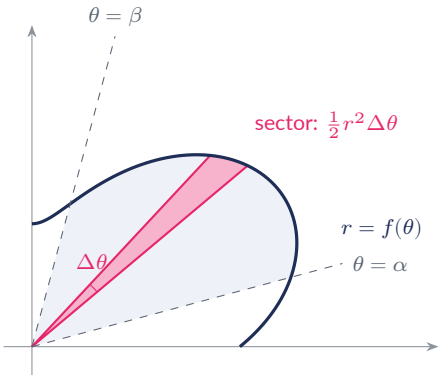
$\frac{dy}{d\theta} = \sqrt{3} + 2 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$.
$\frac{dy}{dt} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dt} = \frac{3\sqrt{3}}{2} \cdot 3 = \boxed{\frac{9\sqrt{3}}{2}}$. The particle's distance from the origin and its y-coordinate are both increasing at that instant.

(!) Lỗi thường gặp: Không được viết hệ số góc là $\frac{dr}{d\theta}$ hay $\frac{r' \sin\theta}{r' \cos\theta}$ (quên số hạng $r\cos\theta$ và $-r\sin\theta$ của quy tắc tích). Luôn đi qua $x = r\cos\theta$, $y = r\sin\theta$ rồi đạo hàm theo θ bằng quy tắc tích. Máy tính phải để ở chế độ **radian**.

9.8 Finding the Area of a Polar Region or the Area Bounded by a Single Polar Curve

For polar regions the natural "slice" is not a thin rectangle but a thin **circular sector**. A sector of radius r and central angle $\Delta\theta$ has area $\frac{1}{2}r^2\Delta\theta$. Slicing the region between the rays $\theta = \alpha$ and $\theta = \beta$ into thin sectors and summing:

$$A \approx \sum \tfrac{1}{2}\big[f(\theta_i)\big]^2 \Delta\theta \longrightarrow \int_\alpha^\beta \tfrac{1}{2}\big[f(\theta)\big]^2\, d\theta.$$



The region between $\theta = \alpha$, $\theta = \beta$ and the curve $r = f(\theta)$ is approximated by thin circular sectors of area $\frac{1}{2}r^2\Delta\theta$.

Area of a polar region

If f is continuous and the region is swept out exactly once as θ goes from α to β ($\beta - \alpha \le 2\pi$), then

$$A = \frac{1}{2}\int_\alpha^\beta r^2\, d\theta = \frac{1}{2}\int_\alpha^\beta \big[f(\theta)\big]^2\, d\theta.$$

Useful identities: $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$, $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$.

VN GIẢI THÍCH TIẾNG VIỆT

Diện tích miền cực: $A = \frac{1}{2}\int_\alpha^\beta r^2\, d\theta$ — cộng các **hình quạt mỏng** (diện tích $\frac{1}{2}r^2\Delta\theta$), không phải các hình chữ nhật. Ba bước: (1) vẽ phác đường cong; (2) tìm khoảng θ để miền được quét **đúng một lần** (thường tìm các góc có $r = 0$); (3) dùng công thức hạ bậc $\cos^2\theta = \frac{1+\cos 2\theta}{2}$, $\sin^2\theta = \frac{1-\cos 2\theta}{2}$. Tận dụng tính đối xứng để rút gọn cận.

Ví dụ mẫu · Area inside a cardioid

Find the area enclosed by the cardioid $r = 1 + \cos\theta$.

The cardioid is traced once for $0 \le \theta \le 2\pi$ and is symmetric about the x-axis, so

$$A = 2 \cdot \frac{1}{2}\int_0^{\pi} (1+\cos\theta)^2\, d\theta = \int_0^{\pi} \left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \left[\tfrac{3}{2}\theta + 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{\pi} = \boxed{\frac{3\pi}{2}}.$$

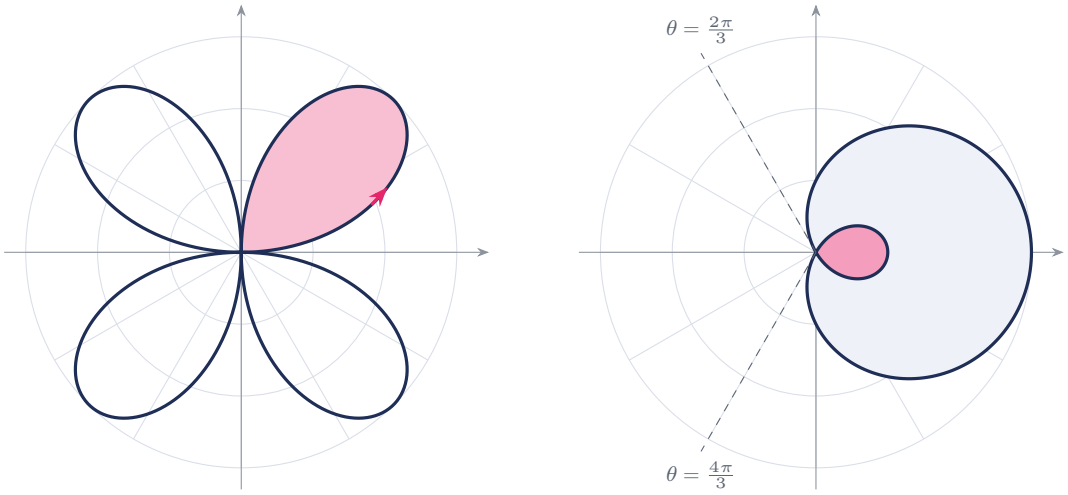
Ví dụ mẫu · One petal of a rose

Find the area of one petal of the rose $r = 3\sin 2\theta$.

Limits: $r = 0$ when $\sin 2\theta = 0$, i.e. $\theta = 0, \frac{\pi}{2}, \pi, \ldots$ For $0 \le \theta \le \frac{\pi}{2}$, $r \ge 0$ and one petal (in Quadrant I) is traced from the pole back to the pole.

$$A = \frac{1}{2}\int_0^{\pi/2} 9\sin^2 2\theta\, d\theta = \frac{9}{2}\int_0^{\pi/2} \frac{1-\cos 4\theta}{2}\, d\theta = \frac{9}{4}\left[\theta - \tfrac{1}{4}\sin 4\theta\right]_0^{\pi/2} = \frac{9}{4}\cdot\frac{\pi}{2} = \boxed{\frac{9\pi}{8}}.$$

(All four petals: $4 \cdot \frac{9\pi}{8} = \frac{9\pi}{2}$, which also equals $\frac{1}{2}\int_0^{2\pi} 9\sin^2 2\theta\, d\theta$.)



one petal of $r = 3\sin 2\theta$, $0 \le \theta \le \frac{\pi}{2}$ — inner loop of $r = 1 + 2\cos\theta$ (pink)

Left: one petal of the four-petal rose. Right: the inner loop of $r = 1 + 2\cos\theta$ is traced for $\frac{2\pi}{3} \le \theta \le \frac{4\pi}{3}$, where $r \le 0$ (the rays are where $r = 0$); the pale navy region lies inside the outer loop but outside the inner loop.

Ví dụ mẫu · The inner loop of a limaçon

For the limaçon $r = 1 + 2\cos\theta$, find (a) the area of the inner loop, and (b) the area of the region inside the outer loop but outside the inner loop.

Where is $r = 0$? $\cos\theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$. For $\frac{2\pi}{3} < \theta < \frac{4\pi}{3}$, $r < 0$: this interval traces the inner loop (in the right half-plane, by the negative-r rule).

Antiderivative: $(1+2\cos\theta)^2 = 1+4\cos\theta+4\cos^2\theta = 3+4\cos\theta+2\cos 2\theta$, so

$$\int (1+2\cos\theta)^2\,d\theta = 3\theta+4\sin\theta+\sin 2\theta.$$

(a) $A_{\text{inner}} = \frac{1}{2}\Big[3\theta+4\sin\theta+\sin 2\theta\Big]_{2\pi/3}^{4\pi/3} = \frac{1}{2}\Big[2\pi+4(-\sqrt{3})+\sqrt{3}\Big] = \boxed{\pi-\frac{3\sqrt{3}}{2}} \approx 0.544.$

(b) The outer loop is traced for $-\frac{2\pi}{3} \le \theta \le \frac{2\pi}{3}$:

$$A_{\text{outer}} = \frac{1}{2}\Big[3\theta+4\sin\theta+\sin 2\theta\Big]_{-2\pi/3}^{2\pi/3} = \frac{1}{2}\Big[4\pi+4\sqrt{3}-\sqrt{3}\Big] = 2\pi+\frac{3\sqrt{3}}{2}.$$

Region between the loops $= A_{\text{outer}} - A_{\text{inner}} = \left(2\pi+\frac{3\sqrt{3}}{2}\right) - \left(\pi-\frac{3\sqrt{3}}{2}\right) = \boxed{\pi+3\sqrt{3}} \approx 8.338.$

Ví dụ mẫu · A circle through the pole

Find the area inside $r = 4\sin\theta$ using a polar integral, and explain why $\frac{1}{2}\int_0^{2\pi} 16\sin^2\theta\,d\theta$ is wrong.

$r = 4\sin\theta$ is a circle of radius 2 (centered at $(0,2)$), traced once for $0 \le \theta \le \pi$:

$$A = \frac{1}{2}\int_0^{\pi} 16\sin^2\theta\,d\theta = 8\int_0^{\pi}\frac{1-\cos 2\theta}{2}\,d\theta = 4\Big[\theta-\tfrac{1}{2}\sin 2\theta\Big]_0^{\pi} = \boxed{4\pi},$$

matching $\pi\cdot 2^2$. On $[\pi, 2\pi]$, $r \le 0$ and the same circle is traced a second time, so integrating over $[0, 2\pi]$ gives 8π — double the true area.

Ví dụ mẫu · Area with a calculator

(Calculator allowed.) Find the area of the region enclosed by the curve $r = 2+\sin(3\theta)$ that lies in the first quadrant.

$r > 0$ for all θ, and the first-quadrant part is swept by $0 \le \theta \le \frac{\pi}{2}$:

$$A = \frac{1}{2}\int_0^{\pi/2} (2+\sin 3\theta)^2\,d\theta \approx \boxed{4.201}.$$

(!) Lỗi thường gặp: Ba lỗi kinh điển: (1) quên hệ số $\frac{1}{2}$; (2) viết $\frac{1}{2}\int r\,d\theta$ hoặc $\int r^2\,d\theta$ — phải là r^2 *và* $\frac{1}{2}$; (3) chọn cận làm miền bị quét hai lần (đường tròn qua gốc trên $[0,2\pi]$, hoa hồng n lẻ trên $[0,2\pi]$) hoặc chỉ dùng $[0,2\pi]$ "theo thói quen" khi đề hỏi một phần của miền.

9.9 Finding the Area of the Region Bounded by Two Polar Curves

Area between two polar curves

If $0 \le r_{\text{in}}(\theta) \le r_{\text{out}}(\theta)$ for $\alpha \le \theta \le \beta$, the area of the region between the curves (and between

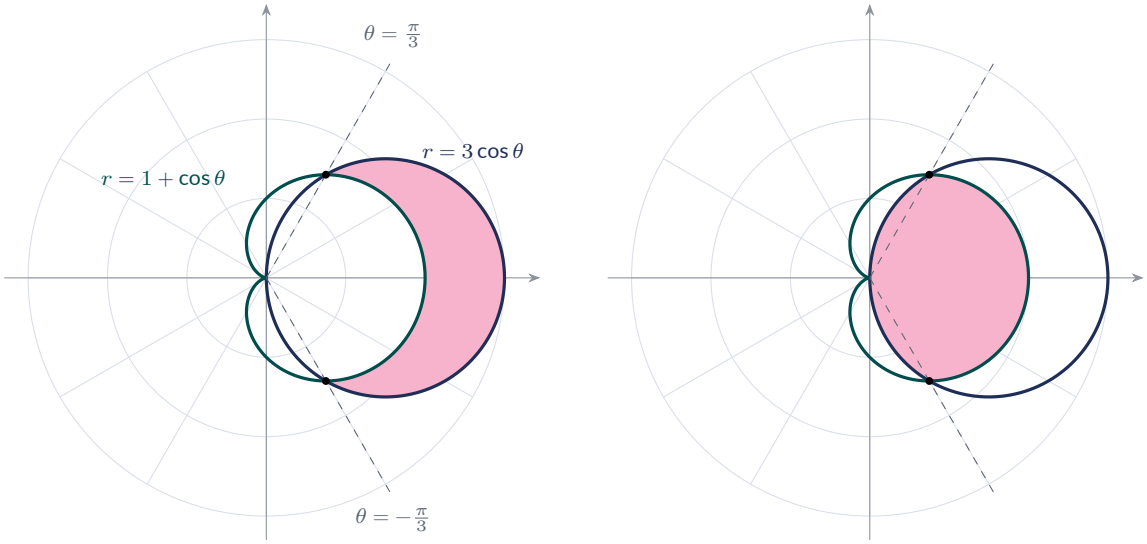
the rays $\theta = \alpha$, $\theta = \beta$) is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left(\left[r_{\text{out}}(\theta)\right]^2 - \left[r_{\text{in}}(\theta)\right]^2 \right) d\theta.$$

Procedure. (1) Sketch both curves. (2) Find intersection points by solving $r_1(\theta) = r_2(\theta)$ — *and separately check the pole*, since the curves may pass through the origin at different values of θ. (3) Decide which curve is outer on each θ-interval; split the integral where the roles change. (4) Square each radius *before* subtracting.

VN **GIẢI THÍCH TIẾNG VIỆT**

Diện tích giữa hai đường cực: $\frac{1}{2}\int (r_{\text{ngoài}}^2 - r_{\text{trong}}^2)\, d\theta$ — **bình phương từng bán kính rồi mới trừ**, *không* phải $\frac{1}{2}\int (r_{\text{ngoài}} - r_{\text{trong}})^2\, d\theta$. Tìm giao điểm bằng cách giải $r_1 = r_2$, nhưng **phải kiểm tra riêng gốc toạ độ**: hai đường có thể cùng đi qua gốc tại các góc θ khác nhau, nên phương trình $r_1 = r_2$ không phát hiện được. Với *miền chung* (vùng nằm trong cả hai đường), thường phải **tách tích phân**: trên mỗi khoảng θ, dùng đường nằm *trong*.



inside the circle, outside the cardioid — common region (inside both)

The circle $r = 3\cos\theta$ and the cardioid $r = 1 + \cos\theta$ meet at $\theta = \pm\frac{\pi}{3}$ (and at the pole). Left: region inside the circle and outside the cardioid. Right: the common region — bounded by the cardioid for $|\theta| \le \frac{\pi}{3}$ and by the circle for $\frac{\pi}{3} \le |\theta| \le \frac{\pi}{2}$.

Ví dụ mẫu · Inside one curve, outside another

Find the area of the region inside the circle $r = 3\cos\theta$ and outside the cardioid $r = 1 + \cos\theta$.

Intersections: $3\cos\theta = 1 + \cos\theta \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \pm\frac{\pi}{3}$, at $r = \frac{3}{2}$.

Outer / inner: at $\theta = 0$, the circle has $r = 3$ and the cardioid $r = 2$, so for $|\theta| < \frac{\pi}{3}$ the circle is outside. Using symmetry about the x-axis:

$$A = 2 \cdot \frac{1}{2}\int_0^{\pi/3} \left[(3\cos\theta)^2 - (1+\cos\theta)^2\right] d\theta = \int_0^{\pi/3} (8\cos^2\theta - 2\cos\theta - 1)\, d\theta.$$

With $8\cos^2\theta = 4 + 4\cos 2\theta$:

$$A = \int_0^{\pi/3} (3 + 4\cos 2\theta - 2\cos\theta)\, d\theta = \left[3\theta + 2\sin 2\theta - 2\sin\theta\right]_0^{\pi/3} = \pi + \sqrt{3} - \sqrt{3} = \boxed{\pi}.$$

Ví dụ mẫu · The common region (splitting the integral)

Find the area of the region inside *both* $r = 3\cos\theta$ and $r = 1 + \cos\theta$.

Which curve bounds the region? For $0 \le \theta \le \frac{\pi}{3}$ the cardioid is the inner curve; for $\frac{\pi}{3} \le \theta \le \frac{\pi}{2}$ the circle is inside (at $\theta = \frac{\pi}{2}$ the circle reaches the pole while the cardioid has $r = 1$). By symmetry,

$$A = 2\left[\frac{1}{2}\int_0^{\pi/3}(1+\cos\theta)^2\,d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2} 9\cos^2\theta\,d\theta\right].$$

First integral: $\int_0^{\pi/3}\left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right)d\theta = \frac{\pi}{2} + \sqrt{3} + \frac{\sqrt{3}}{8} = \frac{\pi}{2} + \frac{9\sqrt{3}}{8}$.

Second integral: $\frac{9}{2}\int_{\pi/3}^{\pi/2}(1+\cos 2\theta)\,d\theta = \frac{9}{2}\left[\frac{\pi}{6} + \frac{1}{2}\left(0 - \frac{\sqrt{3}}{2}\right)\right] = \frac{3\pi}{4} - \frac{9\sqrt{3}}{8}$.

$$A = \left(\frac{\pi}{2} + \frac{9\sqrt{3}}{8}\right) + \left(\frac{3\pi}{4} - \frac{9\sqrt{3}}{8}\right) = \boxed{\frac{5\pi}{4}}.$$

Check: circle area $\frac{9\pi}{4}$ = common region $\frac{5\pi}{4}$ + (inside circle, outside cardioid) π. ✓ consistent with the previous example.

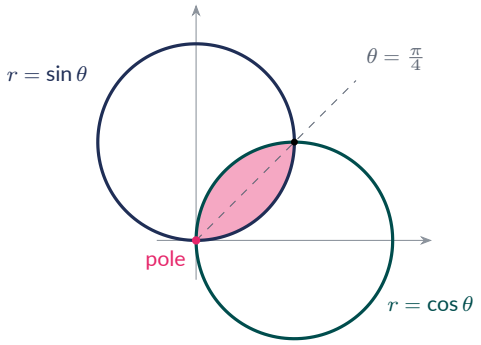
Ví dụ mẫu · Curves that meet at the pole

Find the area of the region common to the circles $r = \sin\theta$ and $r = \cos\theta$.

Intersections. $\sin\theta = \cos\theta$ gives $\theta = \frac{\pi}{4}$, the point $\left(\frac{\sqrt{2}}{2}, \frac{\pi}{4}\right)$. But both curves also pass through the **pole**: $r = \sin\theta$ at $\theta = 0$, and $r = \cos\theta$ at $\theta = \frac{\pi}{2}$. Solving $r_1 = r_2$ cannot detect this, because the curves reach the pole at different angles.

Set-up. For $0 \le \theta \le \frac{\pi}{4}$ the circle $r = \sin\theta$ is inner; for $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$, $r = \cos\theta$ is inner. By symmetry about the line $\theta = \frac{\pi}{4}$:

$$A = 2\cdot\frac{1}{2}\int_0^{\pi/4}\sin^2\theta\,d\theta = \int_0^{\pi/4}\frac{1-\cos 2\theta}{2}\,d\theta = \left[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right]_0^{\pi/4} = \boxed{\frac{\pi}{8} - \frac{1}{4}} = \frac{\pi - 2}{8}.$$



The circles $r = \sin\theta$ and $r = \cos\theta$ meet at $\theta = \frac{\pi}{4}$ and also at the pole (reached at $\theta = 0$ by one curve and $\theta = \frac{\pi}{2}$ by the other). The lens-shaped common region has area $\frac{\pi-2}{8}$.

Ví dụ mẫu · Inside a cardioid, outside a circle

Find the area of the region inside the cardioid $r = 2 + 2\cos\theta$ and outside the circle $r = 3$.

Intersections: $2 + 2\cos\theta = 3 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \pm\frac{\pi}{3}$. For $|\theta| < \frac{\pi}{3}$ the cardioid is outside (at

$\theta = 0$, $r = 4 > 3$).

$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3} \Big[(2+2\cos\theta)^2 - 9\Big]\,d\theta = \frac{1}{2}\int_{-\pi/3}^{\pi/3} \big(-3 + 8\cos\theta + 2\cos 2\theta\big)\,d\theta,$$

since $(2+2\cos\theta)^2 - 9 = 4 + 8\cos\theta + 4\cos^2\theta - 9 = -3 + 8\cos\theta + 2\cos 2\theta$. Then

$$A = \frac{1}{2}\Big[-3\theta + 8\sin\theta + \sin 2\theta\Big]_{-\pi/3}^{\pi/3} = \frac{1}{2}\Big[-2\pi + 8\sqrt{3} + \sqrt{3}\Big] = \boxed{\frac{9\sqrt{3}}{2} - \pi} \approx 4.653.$$

(!) Lỗi thường gặp: Sai lầm nghiêm trọng nhất: viết $\frac{1}{2}\int (r_1 - r_2)^2\,d\theta$. Lỗi thứ hai: chỉ giải $r_1 = r_2$ rồi bỏ sót giao điểm tại gốc (ví dụ $r = \sin\theta$ và $r = \cos\theta$). Lỗi thứ ba: với miền chung, dùng *một* tích phân duy nhất trên cả khoảng trong khi đường bao miền đổi vai tại giao điểm — phải tách tích phân. Luôn vẽ hình trước khi lập tích phân.

9.10 Problem Bank

Parametric equations: derivatives and arc length

Bài tập luyện tập

A curve is defined by $x = t^3 + t$, $y = t^2$. What is $\dfrac{dy}{dx}$ at the point where $t = 1$?

(A) 2

(B) $\dfrac{1}{2}$

(C) $\dfrac{1}{4}$

(D) 4

Lời giải chi tiết

$\dfrac{dx}{dt} = 3t^2 + 1 = 4$ and $\dfrac{dy}{dt} = 2t = 2$ at $t = 1$, so $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2}{4} = \dfrac{1}{2}$. **(B).** Choice (A) inverts the ratio; (D) is $\frac{dx}{dt}$ alone.

Bài tập luyện tập

Which is an equation of the line tangent to the curve $x = e^t$, $y = t^2 + 1$ at the point where $t = 1$?

(A) $y - 2 = 2e(x - e)$

(B) $y - 2 = \dfrac{e}{2}(x - e)$

(C) $y - e = \dfrac{2}{e}(x - 2)$

(D) $y - 2 = \dfrac{2}{e}(x - e)$

Lời giải chi tiết

Point: $x(1) = e$, $y(1) = 2$. Slope: $\dfrac{dy/dt}{dx/dt} = \dfrac{2t}{e^t} = \dfrac{2}{e}$ at $t = 1$. Tangent line $y - 2 = \dfrac{2}{e}(x - e)$. **(D).** (B) uses the reciprocal slope; (C) swaps the coordinates of the point.

Bài tập luyện tập

At which point does the curve $x = t^2 - 4t$, $y = t^3 - 3t$ have a vertical tangent line?

(A) $(-4,\ 2)$
(C) $(5,\ 2)$
(B) $(-3,\ -2)$
(D) $(0,\ 0)$

Lời giải chi tiết

Vertical tangent: $\frac{dx}{dt} = 2t - 4 = 0 \Rightarrow t = 2$, and there $\frac{dy}{dt} = 3(4) - 3 = 9 \neq 0$. Point: $x(2) = 4 - 8 = -4$, $y(2) = 8 - 6 = 2$. **(A)**. Choices (B) and (C) are the points with *horizontal* tangents ($t = 1$ and $t = -1$, where $\frac{dy}{dt} = 0$); (D) is the point at $t = 0$.

Bài tập luyện tập

A curve C is defined by $x = t^2 + 1$ and $y = t^3 - t$.

(a) Find $\dfrac{dy}{dx}$ in terms of t, and write an equation of the line tangent to C at $t = 1$.

(b) Find the coordinates of all points on C at which the tangent line is horizontal.

(c) Find $\dfrac{d^2y}{dx^2}$ in terms of t. Is C concave up or concave down at $t = 1$? Justify.

Lời giải chi tiết

(a) $\frac{dx}{dt} = 2t$, $\frac{dy}{dt} = 3t^2 - 1$, so $\dfrac{dy}{dx} = \dfrac{3t^2 - 1}{2t}$. At $t = 1$: point $(2, 0)$, slope $\dfrac{2}{2} = 1$. Tangent line: $y = x - 2$.

(b) $\frac{dy}{dt} = 0 \Rightarrow t = \pm\frac{1}{\sqrt{3}}$, where $\frac{dx}{dt} = \pm\frac{2}{\sqrt{3}} \neq 0$. Then $x = \frac{1}{3} + 1 = \frac{4}{3}$ and $y = t(t^2 - 1) = \pm\frac{1}{\sqrt{3}}\left(-\frac{2}{3}\right) = \mp\frac{2\sqrt{3}}{9}$. Points: $\left(\frac{4}{3},\ -\frac{2\sqrt{3}}{9}\right)$ (at $t = \frac{1}{\sqrt{3}}$) and $\left(\frac{4}{3},\ \frac{2\sqrt{3}}{9}\right)$ (at $t = -\frac{1}{\sqrt{3}}$).

(c) $\dfrac{dy}{dx} = \dfrac{3}{2}t - \dfrac{1}{2t}$, so $\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) = \dfrac{3}{2} + \dfrac{1}{2t^2} = \dfrac{3t^2 + 1}{2t^2}$. Dividing by $\frac{dx}{dt} = 2t$:

$$\frac{d^2y}{dx^2} = \frac{3t^2 + 1}{4t^3}.$$

At $t = 1$, $\frac{d^2y}{dx^2} = \frac{4}{4} = 1 > 0$, so C is **concave up** at $t = 1$.

Bài tập luyện tập

If $x = \cos t$ and $y = \sin t$, then $\dfrac{d^2y}{dx^2}$ (where defined) equals

(A) $-\csc^2 t$
(C) $-\csc^3 t$
(B) $\csc^3 t$
(D) $\tan t$

Lời giải chi tiết

$\dfrac{dy}{dx} = \dfrac{\cos t}{-\sin t} = -\cot t$. Then $\dfrac{d}{dt}(-\cot t) = \csc^2 t$, and dividing by $\dfrac{dx}{dt} = -\sin t$ gives $\dfrac{d^2y}{dx^2} = \dfrac{\csc^2 t}{-\sin t} = -\csc^3 t$. **(C)**. Choice (A) forgets to divide by $\frac{dx}{dt}$. *Check*: on the upper semicircle

($\sin t > 0$) the result is negative — concave down, as expected.

Bài tập luyện tập

For the curve $x = t^2 + t$, $y = \ln t$, the value of $\dfrac{d^2y}{dx^2}$ at $t = 1$ is

(A) $-\dfrac{5}{27}$ **(C)** $-\dfrac{1}{2}$

(B) $-\dfrac{5}{9}$ **(D)** $\dfrac{1}{3}$

Lời giải chi tiết

$\dfrac{dy}{dx} = \dfrac{1/t}{2t+1} = \dfrac{1}{2t^2+t}$. Then $\dfrac{d}{dt}\left(\dfrac{1}{2t^2+t}\right) = -\dfrac{4t+1}{(2t^2+t)^2}$, which is $-\frac{5}{9}$ at $t = 1$. Divide by $\frac{dx}{dt} = 2t + 1 = 3$: $\dfrac{d^2y}{dx^2} = -\dfrac{5}{27}$. **(A)**. Distractors: (B) forgets to divide by $\frac{dx}{dt}$; (C) is the incorrect $\frac{y''(t)}{x''(t)} = \frac{-1}{2}$; (D) is $\frac{dy}{dx}$ itself.

Bài tập luyện tập

Which integral gives the length of the curve $x = t^2$, $y = \sin t$ for $0 \le t \le \pi$?

(A) $\displaystyle\int_0^{\pi} \sqrt{t^4 + \sin^2 t}\, dt$ **(C)** $\displaystyle\int_0^{\pi} (2t + \cos t)\, dt$

(B) $\displaystyle\int_0^{\pi} \sqrt{1 + \cos^2 t}\, dt$ **(D)** $\displaystyle\int_0^{\pi} \sqrt{4t^2 + \cos^2 t}\, dt$

Lời giải chi tiết

$L = \int_0^{\pi} \sqrt{(x')^2 + (y')^2}\, dt$ with $x' = 2t$, $y' = \cos t$: $\displaystyle\int_0^{\pi} \sqrt{4t^2 + \cos^2 t}\, dt$. **(D)**. (A) uses the coordinates instead of their derivatives; (C) drops the square root (a sum of derivatives is not a length); (B) is the formula for the graph $y = \sin x$, not this curve.

Bài tập luyện tập

Find the exact length of the curve $x = 3t^2$, $y = 2t^3$ for $0 \le t \le 1$.

Lời giải chi tiết

$x' = 6t$, $y' = 6t^2$, so $(x')^2 + (y')^2 = 36t^2 + 36t^4 = 36t^2(1 + t^2)$ and, for $t \ge 0$, $\sqrt{\cdot} = 6t\sqrt{1 + t^2}$.

$$L = \int_0^1 6t\sqrt{1+t^2}\, dt = 3\int_1^2 u^{1/2}\, du \;(u = 1+t^2) = 2\Big[u^{3/2}\Big]_1^2 = 2(2\sqrt{2}-1) = 4\sqrt{2}-2 \approx 3.657.$$

Bài tập luyện tập

(Calculator.) What is the length of the path traced by $x = \sin 2t$, $y = t^2$ for $0 \le t \le 2$?

(A) 5.147
(C) 4.397
(B) 4.071
(D) 6.757

Lời giải chi tiết

$L = \int_0^2 \sqrt{(2\cos 2t)^2 + (2t)^2}\,dt \approx 5.147$. **(A).** (B) is the straight-line distance between the endpoints $(0,0)$ and $(\sin 4, 4)$; (C) forgets the chain-rule factor 2 in $\frac{d}{dt}\sin 2t$; (D) adds $|x'| + |y'|$ instead of using the Pythagorean form.

Bài tập luyện tập

A curve is given by $x = 1 + 2\cos t$, $y = -1 + 2\sin t$ for $0 \le t \le \pi$.

(a) Eliminate the parameter, describe the curve, and give its initial point, terminal point, and orientation.

(b) Find $\dfrac{dy}{dx}$ at $t = \frac{\pi}{4}$.

(c) Set up and evaluate an integral for the length of the curve.

Lời giải chi tiết

(a) $\cos t = \frac{x-1}{2}$, $\sin t = \frac{y+1}{2}$, so $(x-1)^2 + (y+1)^2 = 4$: the circle of radius 2 centered at $(1,-1)$. Since $\sin t \ge 0$ on $[0,\pi]$, $y \ge -1$: only the **upper half** is traced. Initial point ($t = 0$): $(3,-1)$; terminal point ($t = \pi$): $(-1,-1)$; orientation **counterclockwise**.

(b) $\dfrac{dy}{dx} = \dfrac{2\cos t}{-2\sin t} = -\cot t$; at $t = \frac{\pi}{4}$, $\dfrac{dy}{dx} = -1$.

(c) $L = \int_0^{\pi} \sqrt{4\sin^2 t + 4\cos^2 t}\,dt = \int_0^{\pi} 2\,dt = 2\pi$, half the circumference 4π — as expected.

Vector-valued functions and planar motion

Bài tập luyện tập

A particle moves with position $\vec{r}(t) = \langle \sin(\pi t),\ t^3 \rangle$. What is the speed of the particle at $t = 1$?

(A) $\pi + 3$
(C) $\sqrt{\pi^2 + 9}$
(B) 3
(D) $\sqrt{\pi^2 + 1}$

Lời giải chi tiết

$\vec{v}(t) = \langle \pi\cos(\pi t),\ 3t^2 \rangle$, so $\vec{v}(1) = \langle -\pi,\ 3 \rangle$ and speed $= \sqrt{(-\pi)^2 + 3^2} = \sqrt{\pi^2 + 9} \approx 4.344$. **(C).** (A) adds the components; (B) uses only the vertical component; (D) uses the position's y-component $t^3 = 1$ instead of the velocity's.

Bài tập luyện tập

A particle has velocity $\vec{v}(t) = \langle 3t^2,\ 4e^{2t} \rangle$ and is at $\langle 2, -1 \rangle$ when $t = 0$. Where is the particle at $t = 1$?

(A) $\langle 3,\ 2e^2-1\rangle$
(B) $\langle 3,\ 2e^2-3\rangle$
(C) $\langle 1,\ 2e^2\rangle$
(D) $\langle 6,\ 8e^2\rangle$

Lời giải chi tiết

$\vec{r}(t)=\langle t^3+C_1,\ 2e^{2t}+C_2\rangle$. At $t=0$: $C_1=2$ and $2+C_2=-1\Rightarrow C_2=-3$. So $\vec{r}(1)=\langle 3,\ 2e^2-3\rangle$. **(B)**. (A) wrongly takes $C_2=-1$ (forgetting that $2e^0=2$); (C) ignores the initial position; (D) differentiates instead of integrating.

Bài tập luyện tập

A particle has acceleration $\vec{a}(t)=\langle 6t,\ 2\rangle$ and velocity $\vec{v}(1)=\langle 1,\ 0\rangle$. What is the speed of the particle at $t=0$?

(A) 2
(B) 0
(C) $2\sqrt{2}$
(D) $\sqrt{2}$

Lời giải chi tiết

$\vec{v}(t)=\langle 3t^2+C_1,\ 2t+C_2\rangle$; $\vec{v}(1)=\langle 3+C_1,\ 2+C_2\rangle=\langle 1,0\rangle\Rightarrow C_1=-2,\ C_2=-2$. So $\vec{v}(t)=\langle 3t^2-2,\ 2t-2\rangle$, $\vec{v}(0)=\langle -2,-2\rangle$, and speed $=\sqrt{4+4}=2\sqrt{2}$. **(C)**.

Bài tập luyện tập

A particle moves along the ellipse $x=3\cos t$, $y=4\sin t$. What is the maximum speed of the particle?

(A) 3
(B) 4
(C) 5
(D) 7

Lời giải chi tiết

$\vec{v}=\langle -3\sin t,\ 4\cos t\rangle$, so speed$^2=9\sin^2 t+16\cos^2 t=9+7\cos^2 t$, which is largest (16) when $\cos^2 t=1$. Maximum speed $=4$, at the ends of the minor axis, $(\pm 3,0)$. **(B)**. (C) wrongly computes $\sqrt{3^2+4^2}$, which would need $|\sin t|=|\cos t|=1$ simultaneously.

Bài tập luyện tập

A particle moves with $x(t)=t^2-2t$ and $y(t)=t^3-12t$. At what time(s) is the particle at rest?

(A) $t=1$ only
(B) $t=2$ only
(C) $t=1$ and $t=2$
(D) The particle is never at rest.

Lời giải chi tiết

At rest requires $x'(t)=2t-2=0$ *and* $y'(t)=3t^2-12=0$ simultaneously. The first holds only at $t=1$, the second only at $t=\pm 2$. There is no common time: **(D)**. At $t=1$ the particle moves vertically; at $t=\pm 2$ it moves horizontally.

Bài tập luyện tập

(Calculator.) A particle moves with velocity $\vec{v}(t) = \langle \cos(t^2),\ \sin t + 1 \rangle$. What is the total distance traveled by the particle for $0 \le t \le 2$?

(A) 3.447
(C) 3.416
(B) 4.910
(D) 3.798

Lời giải chi tiết

Distance $= \displaystyle\int_0^2 \sqrt{\cos^2(t^2) + (\sin t + 1)^2}\, dt \approx 3.798$. **(D).** (A) is the magnitude of the displacement vector $\left\langle \int_0^2 \cos(t^2)\, dt, \int_0^2 (\sin t + 1)\, dt \right\rangle \approx \langle 0.461, 3.416 \rangle$; (C) is only the vertical displacement; (B) integrates $|x'| + |y'|$.

Bài tập luyện tập

(Calculator.) The x-coordinate of a moving particle satisfies $x'(t) = \sqrt{t^3 + 1}$ and $x(0) = 3$. What is $x(2)$?

(A) 6.241
(C) 9.000
(B) 3.241
(D) 5.000

Lời giải chi tiết

$x(2) = x(0) + \displaystyle\int_0^2 \sqrt{t^3 + 1}\, dt = 3 + 3.241\ldots = 6.241$. **(A).** (B) forgets the initial position; (C) is the tangent-line estimate $3 + x'(2) \cdot 2$; (D) wrongly computes $3 + x'(2) - x'(0)$.

Bài tập luyện tập

(No calculator.) A particle moves in the plane with $x(t) = t^3 - 12t$ and $y(t) = 3t^2 - 12t$ for $t \ge 0$.

(a) Find the velocity vector and the speed at $t = 1$.

(b) Find all times at which the particle is at rest, and the intervals on which it moves to the left.

(c) Is the speed of the particle increasing or decreasing at $t = 1$? Justify.

(d) Find the displacement vector from $t = 0$ to $t = 3$.

(e) Find $\displaystyle\lim_{t \to 2} \frac{y'(t)}{x'(t)}$ and interpret it.

Lời giải chi tiết

$\vec{v}(t) = \langle 3t^2 - 12,\ 6t - 12 \rangle = \langle 3(t-2)(t+2),\ 6(t-2) \rangle$, $\vec{a}(t) = \langle 6t,\ 6 \rangle$.

(a) $\vec{v}(1) = \langle -9,\ -6 \rangle$; speed $= \sqrt{81 + 36} = \sqrt{117} = 3\sqrt{13} \approx 10.817$.

(b) Both components vanish only at $t = 2$ (for $t \ge 0$): the particle is at rest at $t = 2$. Moving left: $x'(t) < 0 \iff 0 \le t < 2$.

(c) $\vec{a}(1) = \langle 6, 6 \rangle$. $x'x'' + y'y'' = (-9)(6) + (-6)(6) = -90 < 0$, so $\frac{d}{dt}(\text{speed}) = \frac{-90}{3\sqrt{13}} < 0$: the speed is **decreasing** at $t = 1$ (velocity and acceleration point in "opposing" directions).

(d) $\langle x(3)-x(0),\ y(3)-y(0)\rangle = \langle 27-36,\ 27-36\rangle = \langle -9,\ -9\rangle$.

(e) For $t \neq 2$, $\dfrac{y'(t)}{x'(t)} = \dfrac{6(t-2)}{3(t-2)(t+2)} = \dfrac{2}{t+2} \to \dfrac{1}{2}$. Although $\frac{dy}{dx}$ is undefined at the instant $t = 2$ (the particle stops), the path has a well-defined tangent line of slope $\frac{1}{2}$ at the point $(x(2), y(2)) = (-16, -12)$.

Bài tập luyện tập

(Calculator.) A particle moves in the xy-plane so that for $0 \le t \le 4$,

$$\frac{dx}{dt} = \ln(1+t^2) - 1, \qquad \frac{dy}{dt} = 3e^{-0.5t}.$$

At $t = 0$ the particle is at $(1, 2)$.

(a) For what values of t, $0 \le t \le 4$, is the particle moving to the left?

(b) Find the speed of the particle and the slope of its path at $t = 2$.

(c) Find the total distance traveled by the particle for $0 \le t \le 4$.

(d) Find the position of the particle at $t = 4$.

(e) Find the minimum x-coordinate of the particle for $0 \le t \le 4$. Justify.

Lời giải chi tiết

(a) $x'(t) < 0 \iff \ln(1+t^2) < 1 \iff t^2 < e-1 \iff t < \sqrt{e-1} \approx 1.311$. Moving left for $0 \le t < 1.311$.

(b) $x'(2) = \ln 5 - 1 = 0.6094\ldots$, $y'(2) = 3e^{-1} = 1.1036\ldots$ Speed $= \sqrt{(x'(2))^2 + (y'(2))^2} = 1.261$. Slope $= \dfrac{y'(2)}{x'(2)} = 1.811$.

(c) Distance $= \displaystyle\int_0^4 \sqrt{\left(\ln(1+t^2)-1\right)^2 + \left(3e^{-0.5t}\right)^2}\, dt = 6.938$.

(d) $x(4) = 1 + \displaystyle\int_0^4 (\ln(1+t^2)-1)dt = 2.984$; $y(4) = 2 + \displaystyle\int_0^4 3e^{-0.5t}\, dt = 2 + 6(1-e^{-2}) = 7.188$. Position $(2.984,\ 7.188)$.

(e) $x'(t)$ changes from negative to positive only at $t_0 = \sqrt{e-1}$, so on $[0, 4]$ the minimum of x is at t_0 or an endpoint. $x(t_0) = 1 + \displaystyle\int_0^{t_0} (\ln(1+t^2)-1)dt = 0.217$, while $x(0) = 1$ and $x(4) = 2.984$. The minimum x-coordinate is 0.217 (at $t = 1.311$).

Polar coordinates, slopes, and $dr/d\theta$

Bài tập luyện tập

The point with polar coordinates $\left(-2,\ \frac{\pi}{3}\right)$ has rectangular coordinates

(A) $(-1,\ -\sqrt{3})$

(B) $(-\sqrt{3},\ -1)$

(C) $(1,\ \sqrt{3})$

(D) $(1,\ -\sqrt{3})$

Lời giải chi tiết

$x = -2\cos\frac{\pi}{3} = -1$, $y = -2\sin\frac{\pi}{3} = -\sqrt{3}$. **(A)**. The negative radius sends the point into Quadrant III, opposite the direction $\frac{\pi}{3}$; (C) ignores the sign of r, and (B) swaps sin and cos.

Bài tập luyện tập

Which rectangular equation is equivalent to the polar equation $r = 6\cos\theta$?

(A) $x^2 + (y-3)^2 = 9$

(B) $(x-6)^2 + y^2 = 36$

(C) $x^2 + y^2 = 6$

(D) $(x-3)^2 + y^2 = 9$

Lời giải chi tiết

$r^2 = 6r\cos\theta \Rightarrow x^2 + y^2 = 6x \Rightarrow (x-3)^2 + y^2 = 9$, a circle of radius 3 centered at $(3, 0)$. **(D)**. (A) would come from $r = 6\sin\theta$.

Bài tập luyện tập

For the polar curve $r = 2 + 2\sin\theta$, the tangent line at the point where $\theta = \frac{\pi}{6}$ is

(A) horizontal

(B) vertical

(C) of slope $\sqrt{3}$

(D) of slope $\dfrac{1}{\sqrt{3}}$

Lời giải chi tiết

$r = 3$ and $\frac{dr}{d\theta} = 2\cos\theta = \sqrt{3}$ at $\theta = \frac{\pi}{6}$.

$$\frac{dx}{d\theta} = r'\cos\theta - r\sin\theta = \sqrt{3}\cdot\frac{\sqrt{3}}{2} - 3\cdot\frac{1}{2} = 0, \qquad \frac{dy}{d\theta} = r'\sin\theta + r\cos\theta = \frac{\sqrt{3}}{2} + \frac{3\sqrt{3}}{2} = 2\sqrt{3} \neq 0.$$

$\frac{dx}{d\theta} = 0$ with $\frac{dy}{d\theta} \neq 0$: the tangent is **vertical**. **(B)**. (C) mistakes $\frac{dr}{d\theta}$ for the slope.

Bài tập luyện tập

Consider $r = 1 - 2\sin\theta$ for $\frac{\pi}{6} < \theta < \frac{\pi}{2}$. As θ increases on this interval, which statement is true?

(A) The point moves toward the origin, because $\frac{dr}{d\theta} < 0$.

(B) The point moves toward the origin, because $r < 0$.

(C) The point moves away from the origin, because $r < 0$ and $\frac{dr}{d\theta} < 0$.

(D) The point moves away from the origin, because $\frac{dr}{d\theta} > 0$.

Lời giải chi tiết

On $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$, $\sin\theta > \frac{1}{2}$, so $r = 1 - 2\sin\theta < 0$; and $\frac{dr}{d\theta} = -2\cos\theta < 0$. Same sign $\Rightarrow |r|$ is increasing: the point moves **away** from the origin. **(C)**. Choice (A) is the classic trap — a decreasing r means "closer" only when $r > 0$.

Bài tập luyện tập

A particle moves along the polar curve $r = 2 + \cos\theta$ so that $\dfrac{d\theta}{dt} = 2$ for all times t.

(a) Find $\dfrac{dr}{dt}$ at the instant when $\theta = \frac{\pi}{2}$, and interpret its sign.

(b) Find $\dfrac{dx}{dt}$ at the instant when $\theta = \frac{\pi}{2}$, and interpret its sign.

Lời giải chi tiết

(a) $\dfrac{dr}{dt} = \dfrac{dr}{d\theta}\cdot\dfrac{d\theta}{dt} = (-\sin\theta)(2) = -2$ at $\theta = \frac{\pi}{2}$. Since $r = 2 > 0$ and r is decreasing, the particle is moving **closer to the origin** at that instant.

(b) $x = r\cos\theta = 2\cos\theta + \cos^2\theta$, so $\dfrac{dx}{d\theta} = -2\sin\theta - 2\sin\theta\cos\theta = -2$ at $\theta = \frac{\pi}{2}$. Then $\dfrac{dx}{dt} = (-2)(2) = -4 < 0$: the particle is moving **to the left**.

Polar area

Bài tập luyện tập

What is the area of one petal of the rose $r = 2\cos 3\theta$?

(A) $\dfrac{\pi}{3}$

(B) $\dfrac{\pi}{6}$

(C) $\dfrac{2\pi}{3}$

(D) π

Lời giải chi tiết

$r = 0$ when $3\theta = \pm\frac{\pi}{2}$, so one petal is traced for $-\frac{\pi}{6} \le \theta \le \frac{\pi}{6}$:

$$A = \frac{1}{2}\int_{-\pi/6}^{\pi/6} 4\cos^2 3\theta\, d\theta = \int_{-\pi/6}^{\pi/6} (1+\cos 6\theta)\, d\theta = \left[\theta + \tfrac{1}{6}\sin 6\theta\right]_{-\pi/6}^{\pi/6} = \frac{\pi}{3}.$$

(A). (C) results from doubling the limits to $[-\frac{\pi}{3}, \frac{\pi}{3}]$, which sweeps more than one petal; (D) is the area of all three petals.

Bài tập luyện tập

The area of the region enclosed by the convex limaçon $r = 2 + \cos\theta$ is

(A) 4π

(B) 5π

(C) $\dfrac{9\pi}{4}$

(D) $\dfrac{9\pi}{2}$

Lời giải chi tiết

$A = \dfrac{1}{2}\displaystyle\int_0^{2\pi} (4 + 4\cos\theta + \cos^2\theta)\, d\theta = \frac{1}{2}(8\pi + 0 + \pi) = \frac{9\pi}{2}$, using $\displaystyle\int_0^{2\pi}\cos\theta\, d\theta = 0$ and

$\int_0^{2\pi} \cos^2\theta\, d\theta = \pi$. **(D).** (C) is half the correct value (the symmetric half $\frac{1}{2}\int_0^{\pi} r^2\, d\theta$ was not doubled); (A) omits the $\cos^2\theta$ term, giving $\frac{1}{2}(8\pi)$.

Bài tập luyện tập

Consider the limaçon $r = 2 + 4\cos\theta$, $0 \le \theta \le 2\pi$.

(a) Find the values of θ at which the curve passes through the pole.

(b) Find the area of the inner loop.

(c) Find $\frac{dr}{d\theta}$ at $\theta = \frac{5\pi}{6}$. Is the point moving toward or away from the pole there? Explain.

(d) Write an equation for the line tangent to the curve at $\theta = \frac{\pi}{2}$.

Lời giải chi tiết

(a) $2 + 4\cos\theta = 0 \Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3},\ \frac{4\pi}{3}$.

(b) The inner loop corresponds to $\frac{2\pi}{3} \le \theta \le \frac{4\pi}{3}$ (where $r \le 0$). Since $(2 + 4\cos\theta)^2 = 4 + 16\cos\theta + 16\cos^2\theta = 12 + 16\cos\theta + 8\cos 2\theta$,

$$A = \frac{1}{2}\Big[12\theta + 16\sin\theta + 4\sin 2\theta\Big]_{2\pi/3}^{4\pi/3} = \frac{1}{2}\Big[8\pi - 16\sqrt{3} + 4\sqrt{3}\Big] = 4\pi - 6\sqrt{3} \approx 2.174.$$

(c) $\frac{dr}{d\theta} = -4\sin\theta = -2$ at $\theta = \frac{5\pi}{6}$, and $r = 2 + 4\left(-\frac{\sqrt{3}}{2}\right) = 2 - 2\sqrt{3} < 0$. Both negative (same sign), so $|r|$ is increasing: the point is moving **away** from the pole.

(d) At $\theta = \frac{\pi}{2}$: $r = 2$, $r' = -4$, point $(0, 2)$. $\frac{dy}{d\theta} = r'\sin\theta + r\cos\theta = -4$, $\frac{dx}{d\theta} = r'\cos\theta - r\sin\theta = -2$; slope $= 2$. Tangent line: $y = 2x + 2$.

Bài tập luyện tập

What is the area of the region inside both circles $r = 2\sin\theta$ and $r = 2\cos\theta$?

(A) $\frac{\pi}{4}$

(B) $\frac{\pi - 2}{8}$

(C) $\frac{\pi - 2}{2}$

(D) $\pi - 2$

Lời giải chi tiết

The circles meet at $\theta = \frac{\pi}{4}$ and at the pole. By symmetry about $\theta = \frac{\pi}{4}$,

$$A = 2 \cdot \frac{1}{2}\int_0^{\pi/4} 4\sin^2\theta\, d\theta = 2\int_0^{\pi/4} (1 - \cos 2\theta)\, d\theta = 2\Big[\theta - \tfrac{1}{2}\sin 2\theta\Big]_0^{\pi/4} = 2\left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{\pi - 2}{2}.$$

(C). (B) is the answer for the circles of diameter 1 ($r = \sin\theta$, $r = \cos\theta$); scaling lengths by 2 multiplies areas by 4.

Bài tập luyện tập

What is the area of the region inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$?

(A) $\dfrac{\pi}{4}$ **(C)** 2

(B) $2+\dfrac{\pi}{4}$ **(D)** $2+\dfrac{\pi}{2}$

Lời giải chi tiết

Intersections: $1+\cos\theta=1\Rightarrow\theta=\pm\frac{\pi}{2}$; the cardioid is outside for $|\theta|<\frac{\pi}{2}$.

$$A=\frac{1}{2}\int_{-\pi/2}^{\pi/2}\Big[(1+\cos\theta)^2-1\Big]d\theta=\frac{1}{2}\int_{-\pi/2}^{\pi/2}(2\cos\theta+\cos^2\theta)\,d\theta=\frac{1}{2}\left(4+\frac{\pi}{2}\right)=2+\frac{\pi}{4}.$$

(B). (D) uses $\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta=\pi$ instead of the correct $\frac{\pi}{2}$; (A) drops the $2\cos\theta$ term.

Bài tập luyện tập

(No calculator.) Find the area of the region that lies inside the circle $r=2\sin\theta$ and outside the circle $r=1$.

Lời giải chi tiết

Intersections: $2\sin\theta=1\Rightarrow\theta=\frac{\pi}{6},\frac{5\pi}{6}$. For $\frac{\pi}{6}<\theta<\frac{5\pi}{6}$, $2\sin\theta>1$.

$$A=\frac{1}{2}\int_{\pi/6}^{5\pi/6}(4\sin^2\theta-1)\,d\theta=\frac{1}{2}\int_{\pi/6}^{5\pi/6}(1-2\cos2\theta)\,d\theta=\frac{1}{2}\Big[\theta-\sin2\theta\Big]_{\pi/6}^{5\pi/6}.$$

$$=\tfrac{1}{2}\left[\left(\tfrac{5\pi}{6}+\tfrac{\sqrt3}{2}\right)-\left(\tfrac{\pi}{6}-\tfrac{\sqrt3}{2}\right)\right]=\tfrac{1}{2}\left(\tfrac{2\pi}{3}+\sqrt3\right)=\frac{\pi}{3}+\frac{\sqrt3}{2}\approx1.913.$$

Bài tập luyện tập

(No calculator.) Let R be the region inside the circle $r=3\sin\theta$ and outside the cardioid $r=1+\sin\theta$.

(a) Find the polar coordinates of the points of intersection of the two curves other than the pole.

(b) Find the area of R.

(c) Find the area of the region inside both curves.

(d) Show that the cardioid has a vertical tangent line at the intersection point in Quadrant I.

Lời giải chi tiết

(a) $3\sin\theta=1+\sin\theta\Rightarrow\sin\theta=\frac{1}{2}\Rightarrow\theta=\frac{\pi}{6},\frac{5\pi}{6}$, with $r=\frac{3}{2}$: the points $\left(\frac{3}{2},\frac{\pi}{6}\right)$ and $\left(\frac{3}{2},\frac{5\pi}{6}\right)$.
(b) For $\frac{\pi}{6}<\theta<\frac{5\pi}{6}$ the circle is outside (at $\theta=\frac{\pi}{2}$: $3>2$).

$$A_R=\frac{1}{2}\int_{\pi/6}^{5\pi/6}\Big[9\sin^2\theta-(1+\sin\theta)^2\Big]d\theta=\frac{1}{2}\int_{\pi/6}^{5\pi/6}(3-4\cos2\theta-2\sin\theta)\,d\theta,$$

since $8\sin^2\theta-2\sin\theta-1=4-4\cos2\theta-2\sin\theta-1$. Antiderivative $3\theta-2\sin2\theta+2\cos\theta$: at $\frac{5\pi}{6}$ it equals $\frac{5\pi}{2}+\sqrt3-\sqrt3=\frac{5\pi}{2}$; at $\frac{\pi}{6}$ it equals $\frac{\pi}{2}-\sqrt3+\sqrt3=\frac{\pi}{2}$. So $A_R=\frac{1}{2}(2\pi)=\pi$.

(c) The circle $r = 3\sin\theta$ (radius $\frac{3}{2}$) has area $\frac{9\pi}{4}$; it splits into R and the common region. Common region $= \frac{9\pi}{4} - \pi = \frac{5\pi}{4}$. (This is the configuration of Section 9.9 rotated by $\frac{\pi}{2}$.)

(d) For $r = 1 + \sin\theta$ at $\theta = \frac{\pi}{6}$: $r = \frac{3}{2}$, $r' = \cos\theta = \frac{\sqrt{3}}{2}$. $\frac{dx}{d\theta} = r'\cos\theta - r\sin\theta = \frac{3}{4} - \frac{3}{4} = 0$, while $\frac{dy}{d\theta} = r'\sin\theta + r\cos\theta = \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{4} = \sqrt{3} \neq 0$. Hence the tangent line there is vertical.

Bài tập luyện tập

(Calculator.) The curves $r = 4$ and $r = 3 + 2\sin(2\theta)$ are graphed in the xy-plane. Let S be the region in the first quadrant that lies inside $r = 3 + 2\sin(2\theta)$ and outside $r = 4$.

(a) Find the values of θ, $0 \le \theta \le \frac{\pi}{2}$, at which the two curves intersect.

(b) Find the area of S.

(c) Find $\frac{dr}{d\theta}$ for $r = 3 + 2\sin(2\theta)$ at $\theta = \frac{\pi}{3}$, and interpret its meaning in terms of a point moving along the curve.

(d) Find the y-coordinate of the point on $r = 3 + 2\sin(2\theta)$ at $\theta = \frac{\pi}{3}$, and find $\frac{dy}{d\theta}$ there. Interpret the sign.

(e) Find the total area enclosed by $r = 3 + 2\sin(2\theta)$.

Lời giải chi tiết

(a) $3 + 2\sin 2\theta = 4 \Rightarrow \sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{6}, \frac{5\pi}{6} \Rightarrow \theta = \frac{\pi}{12} \approx 0.262$ and $\theta = \frac{5\pi}{12} \approx 1.309$.

(b) Between these angles $3 + 2\sin 2\theta > 4$:

$$\text{Area}(S) = \frac{1}{2}\int_{\pi/12}^{5\pi/12} \left[(3 + 2\sin 2\theta)^2 - 16\right] d\theta \approx 3.011.$$

(c) $\frac{dr}{d\theta} = 4\cos 2\theta = 4\cos\frac{2\pi}{3} = -2$. Since $r = 3 + \sqrt{3} \approx 4.732 > 0$ and $\frac{dr}{d\theta} < 0$, the point is moving **closer to the origin** as θ increases, at 2 units per radian.

(d) $y = r\sin\theta = (3+\sqrt{3})\frac{\sqrt{3}}{2} \approx 4.098$. $\frac{dy}{d\theta} = r'\sin\theta + r\cos\theta = (-2)\frac{\sqrt{3}}{2} + (3+\sqrt{3})\frac{1}{2} \approx 0.634 > 0$: as θ increases through $\frac{\pi}{3}$, the point's y-coordinate is increasing (the point is moving upward).

(e) $r > 0$ for all θ, so $A = \frac{1}{2}\int_0^{2\pi} (3 + 2\sin 2\theta)^2 d\theta \approx 34.558$. (Exactly: $\frac{1}{2}\int_0^{2\pi}(9 + 12\sin 2\theta + 4\sin^2 2\theta)\, d\theta = \frac{1}{2}(18\pi + 4\pi) = 11\pi$.)

Ôn tập nhanh — Unit 9:

- **Tham số:** $dy/dx = y'(t)/x'(t)$; tiếp tuyến ngang: $y' = 0,\ x' \neq 0$; đứng: $x' = 0,\ y' \neq 0$; cả hai bằng 0 thì xét giới hạn.
- **Đạo hàm cấp hai:** $d^2y/dx^2 = \left[\frac{d}{dt}(dy/dx)\right] \Big/ x'(t)$ — nhớ chia cho $x'(t)$ lần nữa, không lấy $y''(t)/x''(t)$.
- **Độ dài cung / quãng đường:** $\int_a^b \sqrt{(x')^2 + (y')^2}\, dt = \int$ tốc độ; kiểm tra đường có bị đi lại hai lần không.
- **Hàm vectơ:** đạo hàm, tích phân từng thành phần; vị trí $=$ vị trí đầu $+ \int$ vận tốc; độ dời là vectơ, quãng đường là số.
- **Toạ độ cực:** $x = r\cos\theta$, $y = r\sin\theta$; hệ số góc $= (dy/d\theta)/(dx/d\theta)$ với quy tắc tích; r và $dr/d\theta$ cùng dấu $\Rightarrow$ ra xa gốc.
- **Diện tích cực:** $\frac{1}{2}\int r^2\, d\theta$; giữa hai đường $\frac{1}{2}\int (r^2_{\text{ngoài}} - r^2_{\text{trong}})\, d\theta$; tìm giao điểm cả bằng $r_1 = r_2$ *lẫn* tại gốc; miền chung thường phải tách tích phân.

Infinite Sequences and Series

Mục tiêu Unit: Phân biệt **dãy số** và **chuỗi số**; hiểu tổng của chuỗi là giới hạn của dãy **tổng riêng phần** S_n, tính tổng chuỗi **telescoping** và chuỗi **hình học** (kể cả số thập phân tuần hoàn, chỉ số bắt đầu khác 0); dùng **phép thử số hạng thứ** n để kết luận phân kỳ; áp dụng đúng điều kiện của **phép thử tích phân**, **chuỗi điều hoà và chuỗi** p, **so sánh trực tiếp** và **so sánh giới hạn**, **chuỗi đan dấu**, **phép thử tỉ số**; phân biệt **hội tụ tuyệt đối** và **hội tụ có điều kiện**; ước lượng sai số bằng **cận sai số chuỗi đan dấu**; biết chọn phép thử phù hợp và trình bày lập luận đủ điều kiện như rubric AP. Phần sau của Unit: **đa thức Taylor** và sai số Lagrange, **chuỗi lũy thừa**, bán kính và khoảng hội tụ (kiểm tra hai đầu mút), chuỗi Taylor/Maclaurin của e^x, $\sin x$, $\cos x$, $\frac{1}{1-x}$, và cách tạo chuỗi mới bằng thay biến, đạo hàm, tích phân từng số hạng.

Units 1–8 studied functions of a real variable: limits, derivatives, integrals. Unit 10 asks a deceptively simple question — *what does it mean to add infinitely many numbers?* — and then builds the tools needed to answer it for a given list of terms. Those tools pay off in the second half of the unit, where functions such as e^x and $\sin x$ are rewritten as "infinite polynomials." On the AP Calculus BC exam this unit carries the largest weight of any unit (about 17–18% of the exam), and nearly every free-response section contains a series question.

10.1 Defining Convergent and Divergent Infinite Series

10.1.1 Sequences versus series

A **sequence** is an ordered list of numbers $a_1, a_2, a_3, \ldots$, written $\{a_n\}$; formally it is a function whose domain is the positive (or nonnegative) integers. A sequence **converges** to L if $\lim_{n\to\infty} a_n = L$ for a finite number L; otherwise it **diverges**. Because $a_n = f(n)$ for some formula f, limits of sequences are computed exactly like limits at infinity in Unit 1 (dominant terms, L'Hospital's Rule applied to $f(x)$, the Squeeze Theorem).

A **series** is the *sum* of the terms of a sequence:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots.$$

Adding infinitely many numbers is not an arithmetic operation — no one can finish it — so the sum is *defined* through a limit.

Partial sums and the sum of a series

The nth **partial sum** of $\sum a_k$ is the finite sum

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

The partial sums form a new sequence $S_1, S_2, S_3, \ldots$. If $\lim_{n\to\infty} S_n = S$ exists (finite), the series **converges** and its **sum** is S:

$$\sum_{n=1}^{\infty} a_n = S \quad \text{means} \quad \lim_{n\to\infty} S_n = S.$$

If $\lim S_n$ does not exist (including $S_n \to \pm\infty$), the series **diverges**.

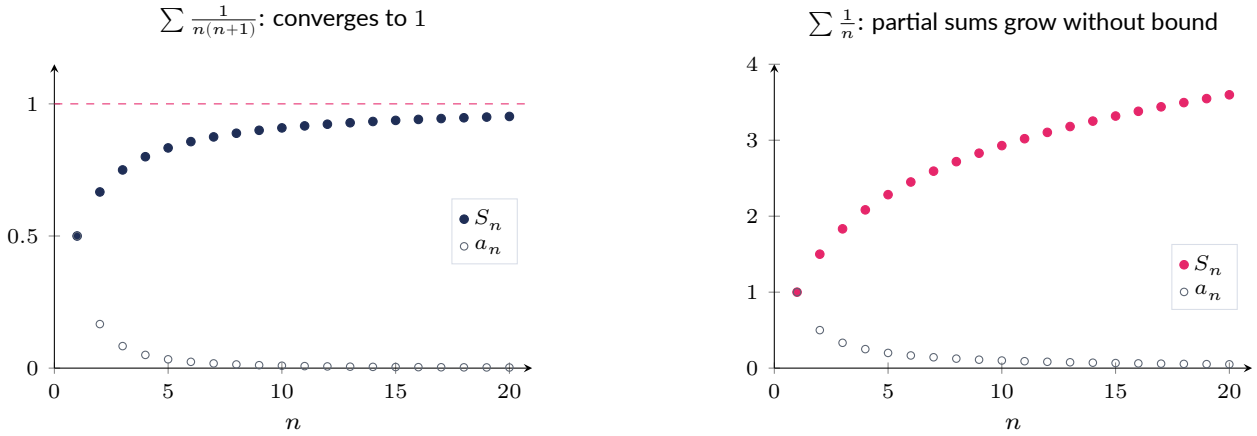
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Dãy số $\{a_n\}$ là một danh sách vô hạn các số; **chuỗi số** $\sum a_n$ là *tổng* các số hạng của dãy. Không ai cộng được vô hạn số, nên tổng của chuỗi được *định nghĩa* qua giới hạn: lập dãy **tổng riêng phần** $S_n = a_1 + \cdots + a_n$; nếu S_n tiến tới một số hữu hạn S thì chuỗi **hội tụ** và có tổng S, ngược lại chuỗi **phân kỳ**. Luôn có *hai* dãy liên quan: dãy số hạng $\{a_n\}$ và dãy tổng riêng phần $\{S_n\}$ — đừng nhầm "dãy a_n hội tụ" với "chuỗi $\sum a_n$ hội tụ".

Two sequences are always attached to a series: the sequence of *terms* $\{a_n\}$ and the sequence of *partial sums* $\{S_n\}$. They behave very differently. If $a_n = \dfrac{n}{n+1}$, the sequence converges ($a_n \to 1$), but every term is at least $\frac{1}{2}$, so $S_n \geq \frac{n}{2} \to \infty$ and the series diverges. Conversely, the partial sums of a convergent series converge to S while the terms themselves converge to 0.

Two facts connect a_n and S_n in both directions:

$$S_n = S_{n-1} + a_n \qquad \text{and} \qquad a_n = S_n - S_{n-1} \quad (n \geq 2), \qquad a_1 = S_1.$$



In both panels the terms a_n (hollow dots) shrink to 0. On the left the partial sums S_n level off at the horizontal asymptote $S = 1$: the series converges. On the right the partial sums of the harmonic series keep climbing (slowly, like $\ln n$): the series diverges even though $a_n \to 0$.

Ví dụ mẫu · Sequence versus series

Let $a_n = \dfrac{3n}{2n+5}$. (a) Does the sequence $\{a_n\}$ converge? (b) Does the series $\sum_{n=1}^{\infty} a_n$ converge?

(a) Divide numerator and denominator by n: $a_n = \dfrac{3}{2 + 5/n} \to \dfrac{3}{2}$. The *sequence* converges to $\frac{3}{2}$.

(b) Because $a_n \to \frac{3}{2}$, eventually every term exceeds 1 (indeed $a_n > 1 \iff 3n > 2n + 5 \iff n > 5$). Adding infinitely many terms larger than 1 makes $S_n \to \infty$, so the *series* diverges. (In Section 10.3 this argument becomes the nth Term Test.)

Ví dụ mẫu · Recovering terms and the sum from S_n

The nth partial sum of a series $\sum_{n=1}^{\infty} a_n$ is $S_n = \dfrac{3n}{n+2}$. (a) Find a_1 and a_5. (b) Find a formula for a_n. (c) Does the series converge? If so, find its sum.

(a) $a_1 = S_1 = \frac{3}{3} = 1$. Also $a_5 = S_5 - S_4 = \dfrac{15}{7} - \dfrac{12}{6} = \dfrac{15}{7} - 2 = \dfrac{1}{7}$.

(b) For $n \geq 2$,

$$a_n = S_n - S_{n-1} = \frac{3n}{n+2} - \frac{3(n-1)}{n+1} = \frac{3n(n+1) - 3(n-1)(n+2)}{(n+1)(n+2)} = \frac{6}{(n+1)(n+2)}.$$

This formula also gives $a_1 = \frac{6}{6} = 1$, so it holds for all $n \geq 1$.

(c) $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{3}{1+2/n} = 3$. The series converges and $\sum_{n=1}^{\infty} \frac{6}{(n+1)(n+2)} = 3$.

10.1.2 Telescoping series

For most series there is no closed formula for S_n. A **telescoping series** is a rare exception: each term splits into a difference $a_n = b_n - b_{n+1}$ (or $b_n - b_{n+k}$), and almost everything cancels in the partial sum, like the sections of a collapsing telescope.

If $a_n = b_n - b_{n+1}$, then

$$S_n = (b_1 - b_2) + (b_2 - b_3) + \cdots + (b_n - b_{n+1}) = b_1 - b_{n+1},$$

so $\sum a_n$ converges if and only if $\lim b_{n+1}$ exists, and then $\sum_{n=1}^{\infty} a_n = b_1 - \lim_{n\to\infty} b_n$.

Ví dụ mẫu · A classic telescoping series

Find the sum of $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Partial fractions: $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. Then

$$S_n = \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \left(\tfrac{1}{3} - \tfrac{1}{4}\right) + \cdots + \left(\tfrac{1}{n} - \tfrac{1}{n+1}\right) = 1 - \frac{1}{n+1}.$$

Since $\lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1$, the series converges and its sum is **1**. (This is the left panel of the figure above.)

Ví dụ mẫu · Telescoping with a gap of two

Find $\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3}$.

Factor and split: $\frac{2}{(n+1)(n+3)} = \frac{1}{n+1} - \frac{1}{n+3}$ (check: $\frac{(n+3)-(n+1)}{(n+1)(n+3)} = \frac{2}{(n+1)(n+3)}$). Write out terms:

$$S_n = \left(\tfrac{1}{2} - \tfrac{1}{4}\right) + \left(\tfrac{1}{3} - \tfrac{1}{5}\right) + \left(\tfrac{1}{4} - \tfrac{1}{6}\right) + \left(\tfrac{1}{5} - \tfrac{1}{7}\right) + \cdots + \left(\tfrac{1}{n+1} - \tfrac{1}{n+3}\right).$$

Each negative part cancels a positive part *two* terms later. Only the first two positive parts and the last two negative parts survive:

$$S_n = \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3} \longrightarrow \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$

The series converges to $\frac{5}{6}$.

Ví dụ mẫu · A telescoping series that diverges

Determine whether $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$ converges.

Write $\ln\frac{n+1}{n} = \ln(n+1) - \ln n$. Then

$$S_n = (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \cdots + (\ln(n+1) - \ln n) = \ln(n+1) - \ln 1 = \ln(n+1).$$

Since $\ln(n+1) \to \infty$, the partial sums are unbounded and the series **diverges** — even though the terms $\ln\frac{n+1}{n} \to \ln 1 = 0$.

(!) Lỗi thường gặp: When a telescoping series has a gap of k, exactly k leading pieces and k trailing pieces survive. Write out at least the first four terms and the last two before cancelling; guessing "only the first and last survive" gives $\frac{1}{2}$ instead of $\frac{5}{6}$ in the second example above.

10.1.3 Algebra of convergent series

If $\sum a_n = A$ and $\sum b_n = B$ both converge and c is a constant, then $\sum(a_n \pm b_n) = A \pm B$ and $\sum c\,a_n = cA$. If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum(a_n + b_n)$ diverges. Finally, **adding or deleting finitely many terms never changes convergence** (it changes the sum, but not whether a sum exists). That is why the convergence tests below only need their hypotheses to hold *eventually*, that is, for all n beyond some index.

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Chuỗi **telescoping** là chuỗi mà mỗi số hạng viết được thành hiệu $b_n - b_{n+1}$ (thường nhờ tách phân thức), nên khi cộng, các số hạng giữa triệt tiêu hết và $S_n = b_1 - b_{n+1}$. Nếu khoảng cách là 2 (ví dụ $\frac{1}{n+1} - \frac{1}{n+3}$) thì còn lại *hai* số hạng đầu và *hai* số hạng cuối. Thêm hay bỏ *hữu hạn* số hạng không làm thay đổi tính hội tụ — vì vậy các điều kiện của phép thử chỉ cần đúng "từ một chỉ số nào đó trở đi".

10.2 Geometric Series

10.2.1 The sum of a geometric series

A **geometric series** has a constant ratio r between consecutive terms:

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots \qquad (a \neq 0).$$

Its partial sums have a closed form. If $S_n = a + ar + \cdots + ar^{n-1}$ (n terms), then $rS_n = ar + ar^2 + \cdots + ar^n$; subtracting, $S_n - rS_n = a - ar^n$, so

$$S_n = \frac{a(1-r^n)}{1-r} \qquad (r \neq 1).$$

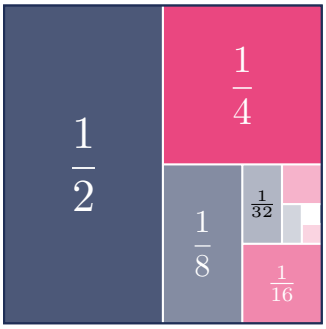
If $|r| < 1$ then $r^n \to 0$ and $S_n \to \dfrac{a}{1-r}$. If $|r| \geq 1$ the terms ar^n do not approach 0 and the series diverges (for $r = 1$, $S_n = na$; for $r = -1$, S_n alternates between a and 0).

Geometric Series Test (CED 10.2)

The geometric series $\sum_{n=0}^{\infty} ar^n$ with $a \neq 0$

- **converges** if $|r| < 1$, and then $\sum_{n=0}^{\infty} ar^n = \dfrac{a}{1-r} = \dfrac{\text{first term}}{1 - \text{ratio}}$;
- **diverges** if $|r| \geq 1$.

The memory device "first term over one minus ratio" works for *any* starting index, as long as "first term" means the actual first term of the series.



Area of the unit square $= 1$.

Each new piece is half of what remains, so the shaded pieces never overflow the square and fill it in the limit:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \frac{1/2}{1 - 1/2} = 1.$$

A geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$. After n pieces the uncovered area is exactly $(\frac{1}{2})^n$, the gap $1 - S_n$, which shrinks to 0.

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Chuỗi **hình học** có tỉ số không đổi r giữa hai số hạng liên tiếp. Chuỗi hội tụ khi và chỉ khi $|r| < 1$, và khi đó tổng bằng **số hạng đầu chia cho** $(1 - r)$. Khi $|r| \geq 1$ chuỗi phân kỳ. Công thức "số hạng đầu / (1 − công bội)" đúng với *mọi* chỉ số bắt đầu — chỉ cần tính đúng số hạng đầu tiên thực sự của chuỗi (ví dụ chuỗi bắt đầu từ $n = 2$ thì số hạng đầu là a_2).

Ví dụ mẫu · Starting at $n = 0$

Find $\sum_{n=0}^{\infty} 5\left(\frac{2}{3}\right)^n$.

The first term ($n = 0$) is 5 and the ratio is $r = \frac{2}{3}$. Since $|r| = \frac{2}{3} < 1$, the series converges, and its sum is $\dfrac{5}{1 - 2/3} = \dfrac{5}{1/3} = \mathbf{15}$.

Ví dụ mẫu · Rewriting into geometric form

Find $\sum_{n=1}^{\infty} \dfrac{3^{n+1}}{4^n}$.

Rewrite each term as $\dfrac{3^{n+1}}{4^n} = 3\left(\dfrac{3}{4}\right)^n$, a geometric series with ratio $r = \frac{3}{4}$, $|r| < 1$, so it converges. The first term ($n = 1$) is $3 \cdot \frac{3}{4} = \frac{9}{4}$. Hence

$$\sum_{n=1}^{\infty} \frac{3^{n+1}}{4^n} = \frac{9/4}{1 - 3/4} = \frac{9/4}{1/4} = \mathbf{9}.$$

Ví dụ mẫu · Negative ratio and a late starting index

Find $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n\,2^n}{5^{n-1}}$.

Write the term as $\dfrac{(-1)^n 2^n}{5^{n-1}} = 5\Big(-\dfrac{2}{5}\Big)^n$, geometric with $r=-\frac{2}{5}$, $|r|<1$: converges. The first term is at $n=2$: $\dfrac{(+1)(4)}{5}=\dfrac{4}{5}$. Therefore

$$\sum_{n=2}^{\infty}\frac{(-1)^n 2^n}{5^{n-1}} = \frac{4/5}{1-(-2/5)} = \frac{4/5}{7/5} = \boxed{\frac{4}{7}}.$$

10.2.2 Repeating decimals

Every repeating decimal is a geometric series in disguise, which is the cleanest proof that it represents a rational number.

Ví dụ mẫu · Repeating decimals as fractions

Write (a) $0.\overline{36}$ and (b) $2.1\overline{45}$ as fractions in lowest terms.

(a) $0.\overline{36} = 0.36 + 0.0036 + 0.000036 + \cdots$, a geometric series with first term $\frac{36}{100}$ and ratio $\frac{1}{100}$:

$$0.\overline{36} = \frac{36/100}{1-1/100} = \frac{36}{99} = \frac{4}{11}.$$

(b) Separate the non-repeating part: $2.1\overline{45} = 2.1 + 0.0\overline{45} = \frac{21}{10} + (0.045 + 0.00045 + \cdots)$. The repeating part has first term $\frac{45}{1000}$ and ratio $\frac{1}{100}$:

$$0.0\overline{45} = \frac{45/1000}{99/100} = \frac{45}{990} = \frac{1}{22}, \qquad 2.1\overline{45} = \frac{21}{10} + \frac{1}{22} = \frac{231+5}{110} = \frac{236}{110} = \frac{118}{55}.$$

Check: $118 \div 55 = 2.14545\ldots$ ✓

10.2.3 Geometric series with a variable ratio

When the ratio contains x, the Geometric Series Test tells us for which x the series converges and what function it adds up to — a preview of power series in the second half of this unit.

Ví dụ mẫu · For which x does it converge?

For what values of x does $\displaystyle\sum_{n=0}^{\infty}\frac{(x-1)^n}{2^n}$ converge? For those x, find the sum as a function of x.

This is geometric with $r=\dfrac{x-1}{2}$ and first term 1. It converges exactly when

$$\left|\frac{x-1}{2}\right| < 1 \iff |x-1|<2 \iff -1<x<3.$$

At $x=-1$ ($r=-1$) and $x=3$ ($r=1$) it diverges. On $-1<x<3$,

$$\sum_{n=0}^{\infty}\Big(\frac{x-1}{2}\Big)^n = \frac{1}{1-\frac{x-1}{2}} = \frac{2}{3-x}.$$

Ví dụ mẫu · The bouncing ball

A ball is dropped from a height of 10 feet. Each time it hits the ground it rebounds to $\frac{3}{4}$ of its previous height. Find the total vertical distance the ball travels.

The ball falls 10 ft. After that, each rebound height is travelled *twice* (up and back down). The rebound heights are $10\cdot\frac{3}{4},\ 10\cdot(\frac{3}{4})^2,\ldots$, a geometric series with first term 7.5 and ratio $\frac{3}{4}$:

$$D = 10 + 2\sum_{n=1}^{\infty} 10\left(\frac{3}{4}\right)^n = 10 + 2\cdot\frac{7.5}{1-3/4} = 10 + 2(30) = \textbf{70 feet}.$$

(!) Lỗi thường gặp: Using $\dfrac{a}{1-r}$ with a equal to the coefficient instead of the first term. In $\sum_{n=1}^{\infty} 3(\frac{3}{4})^n$ the coefficient is 3, but the first term is $\frac{9}{4}$; using 3 gives 12 instead of 9. Also: a geometric series with $|r|\geq 1$ has no sum — never write $\dfrac{a}{1-r}$ for $r=2$.

10.3 The nth Term Test for Divergence

If $\sum a_n$ converges to S, then both $S_n \to S$ and $S_{n-1}\to S$, so

$$a_n = S_n - S_{n-1} \longrightarrow S - S = 0.$$

In words: *the terms of a convergent series must shrink to zero.* The contrapositive is a test for divergence.

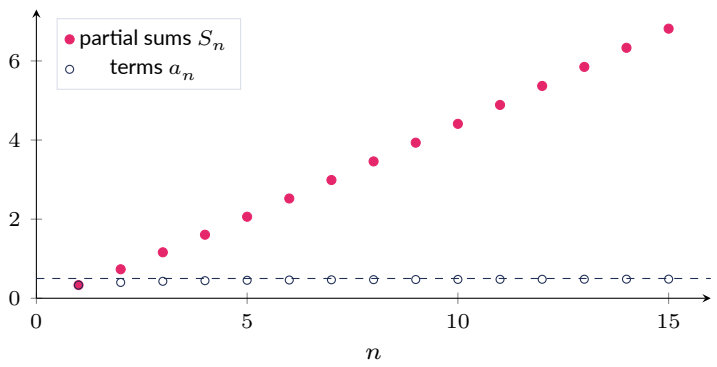
nth Term Test for Divergence (CED 10.3)

If $\lim\limits_{n\to\infty} a_n \neq 0$ (either the limit is a nonzero number or the limit does not exist), then $\sum a_n$ **diverges**.

If $\lim\limits_{n\to\infty} a_n = 0$, the test is **inconclusive**: the series may converge or diverge, and another test is needed.

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Phép thử số hạng thứ n chỉ dùng để chứng minh **phân kỳ**: nếu a_n không tiến về 0 (tiến về số khác 0 hoặc không có giới hạn) thì chuỗi phân kỳ. Nếu $a_n\to 0$ thì *chưa kết luận được gì* — chuỗi điều hoà $\sum\frac{1}{n}$ có $a_n\to 0$ nhưng vẫn phân kỳ. Trên bài thi AP, viết "vì $a_n\to 0$ nên chuỗi hội tụ" là sai và mất điểm lập luận.



Terms $a_n=\frac{n}{2n+1}$ (hollow dots) and partial sums S_n (filled dots). When the terms settle at a nonzero value ($\frac{1}{2}$ here), each new term adds about $\frac{1}{2}$ to the running total, so the partial sums rise without bound.

Ví dụ mẫu · Applying the nth Term Test

Determine whether each series diverges by the nth Term Test.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n+1}$ (b) $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$ (c) $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+1}$ (e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

(a) $\lim\limits_{n\to\infty} \dfrac{n}{2n+1} = \dfrac{1}{2} \neq 0$, so the series diverges by the nth Term Test.
(b) As $n \to \infty$, $\frac{1}{n} \to 0$ and cos is continuous, so $\cos(1/n) \to \cos 0 = 1 \neq 0$: diverges.
(c) $\lim\limits_{n\to\infty} \left(1+\dfrac{1}{n}\right)^n = e \neq 0$: diverges.
(d) $|a_n| = \frac{n}{n+1} \to 1$, so the terms alternate close to $+1$ and -1; $\lim a_n$ does not exist, hence $\lim a_n \neq 0$: diverges.
(e) $\lim \frac{1}{n} = 0$, so the nth Term Test is *inconclusive*. (We show in Section 10.5 that this series diverges, by a different argument.)

Ví dụ mẫu · Using L'Hospital's Rule inside the test

Does $\displaystyle\sum_{n=1}^{\infty} n \sin\left(\frac{2}{n}\right)$ converge?

Let $f(x) = x\sin(2/x) = \dfrac{\sin(2/x)}{1/x}$, a $\frac{0}{0}$ form as $x \to \infty$. By L'Hospital's Rule,

$$\lim_{x\to\infty} \frac{\sin(2/x)}{1/x} = \lim_{x\to\infty} \frac{\cos(2/x)\cdot(-2/x^2)}{-1/x^2} = \lim_{x\to\infty} 2\cos(2/x) = 2.$$

So $a_n \to 2 \neq 0$, and the series diverges by the nth Term Test.

(!) Lỗi thường gặp: The nth Term Test can *never* prove convergence. "$\lim a_n = 0$, therefore $\sum a_n$ converges" is the single most common logical error on series free-response questions. Also, do not confuse the limit of the terms with the limit of the partial sums: for a convergent series, $\lim a_n = 0$ while $\lim S_n = S$.

10.4 The Integral Test

When the terms come from a function that can be integrated, the area under that function and the sum of the terms rise or fall together. Think of each term $a_n = f(n)$ as the area of a rectangle of width 1 and height $f(n)$.

Integral Test (CED 10.4)

Suppose $a_n = f(n)$, where f is **positive**, **continuous**, and **decreasing** on $[k, \infty)$ for some $k \geq 1$. Then

$$\sum_{n=k}^{\infty} a_n \quad \text{and} \quad \int_k^{\infty} f(x)\,dx$$

either **both converge** or **both diverge**.

The integral's value is *not* the sum of the series; the test compares only convergence behaviour.

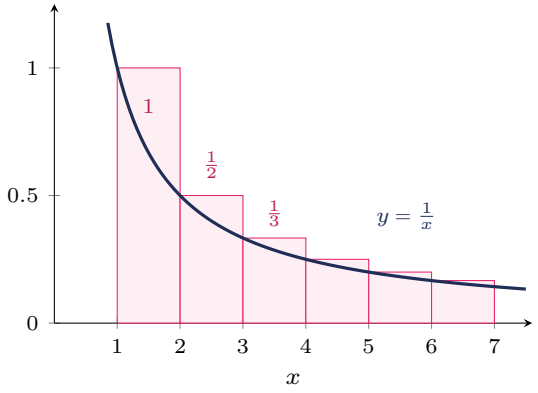
10.4.1 Why it works

Because f is decreasing, on each interval $[n, n+1]$ the rectangle of height $f(n)$ (left endpoint) lies *above* the graph and the rectangle of height $f(n+1)$ (right endpoint) lies *below* it. Adding these comparisons gives

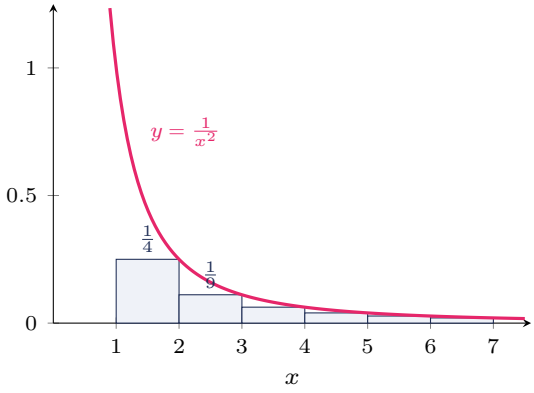
$$\sum_{n=2}^{N} f(n) \;\le\; \int_{1}^{N} f(x)\,dx \;\le\; \sum_{n=1}^{N-1} f(n).$$

If the integral is finite, the left inequality traps the increasing partial sums under a fixed ceiling, so they converge. If the integral is infinite, the right inequality pushes the partial sums to infinity.

Upper rectangles: $\sum_{n=1}^{6} \frac{1}{n} \ge \int_1^7 \frac{dx}{x}$



Lower rectangles: $\sum_{n=2}^{7} \frac{1}{n^2} \le \int_1^7 \frac{dx}{x^2}$



Left: each rectangle of height $\frac{1}{n}$ on $[n, n+1]$ covers the area under $y = \frac{1}{x}$, so $S_N \ge \ln(N+1) \to \infty$ — the harmonic series diverges. Right: each rectangle of height $\frac{1}{n^2}$ on $[n-1, n]$ fits under $y = \frac{1}{x^2}$, so $\sum_{n=2}^{N} \frac{1}{n^2} \le \int_1^{\infty} \frac{dx}{x^2} = 1$ for every N — the partial sums are bounded and the series converges (with $\sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + 1 = 2$).

VN **GIẢI THÍCH TIẾNG VIỆT**

Phép thử tích phân: nếu $a_n = f(n)$ với f **dương, liên tục, giảm** trên $[k, \infty)$ thì chuỗi $\sum_{n=k}^{\infty} a_n$ và tích phân suy rộng $\int_k^{\infty} f(x)\,dx$ *cùng hội tụ* hoặc *cùng phân kỳ*. Hình chữ nhật cao $f(n)$, rộng 1 nằm trên hoặc dưới đồ thị, nên tổng và diện tích bị "kẹp" vào nhau. Trên bài FRQ phải **nêu đủ ba điều kiện** (thường chứng minh "giảm" bằng $f'(x) < 0$). Lưu ý: giá trị tích phân *không* phải là tổng của chuỗi.

Ví dụ mẫu · Integral Test, convergent case

Use the Integral Test to determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges.

Let $f(x) = \dfrac{1}{x^2+1}$. On $[1, \infty)$: $f(x) > 0$; f is continuous (the denominator is never 0); and $f'(x) = \dfrac{-2x}{(x^2+1)^2} < 0$ for $x \ge 1$, so f is decreasing. The hypotheses hold.

$$\int_1^{\infty} \frac{dx}{x^2+1} = \lim_{b\to\infty} \Big[\arctan x\Big]_1^b = \lim_{b\to\infty} \Big(\arctan b - \frac{\pi}{4}\Big) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

The integral converges, so by the Integral Test $\sum \frac{1}{n^2+1}$ **converges**. (Its sum is *not* $\frac{\pi}{4}$; in fact it is about 1.077.)

Ví dụ mẫu · Checking "decreasing" with the derivative

Determine whether $\sum_{n=1}^{\infty} ne^{-n^2}$ converges.

Let $f(x) = xe^{-x^2}$, which is positive and continuous on $[1, \infty)$. Its derivative is

$$f'(x) = e^{-x^2} + x(-2xe^{-x^2}) = e^{-x^2}(1 - 2x^2) < 0 \quad \text{for } x \geq 1,$$

so f is decreasing on $[1, \infty)$. With the substitution $u = x^2$,

$$\int_1^{\infty} xe^{-x^2}\,dx = \lim_{b\to\infty}\left[-\tfrac{1}{2}e^{-x^2}\right]_1^b = \lim_{b\to\infty}\left(-\tfrac{1}{2}e^{-b^2} + \tfrac{1}{2}e^{-1}\right) = \frac{1}{2e}.$$

The integral converges, so the series **converges** by the Integral Test.

Ví dụ mẫu · Integral Test, divergent case

Determine whether $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converges.

Let $f(x) = \frac{1}{x \ln x}$ on $[2, \infty)$. It is positive (both x and $\ln x$ are positive for $x \geq 2$) and continuous. It is decreasing because the denominator $x \ln x$ is a product of two positive increasing functions, hence increasing. With $u = \ln x$, $du = \frac{dx}{x}$:

$$\int_2^{\infty} \frac{dx}{x \ln x} = \lim_{b\to\infty}\Big[\ln(\ln x)\Big]_2^b = \lim_{b\to\infty}\big(\ln(\ln b) - \ln(\ln 2)\big) = \infty.$$

The integral diverges, so the series **diverges** by the Integral Test.

Ví dụ mẫu · "Eventually decreasing" is enough

Determine whether $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges.

Let $f(x) = \frac{\ln x}{x}$. Then $f'(x) = \frac{1 - \ln x}{x^2}$, which is negative for $x > e$. So f is positive, continuous, and decreasing on $[3, \infty)$ (it increases on $[1, e]$, but finitely many terms do not affect convergence). Apply the test on $[3, \infty)$ with $u = \ln x$:

$$\int_3^{\infty} \frac{\ln x}{x}\,dx = \lim_{b\to\infty}\left[\tfrac{1}{2}(\ln x)^2\right]_3^b = \infty.$$

So $\sum_{n=3}^{\infty} \frac{\ln n}{n}$ diverges, and adding the two terms $n = 1, 2$ does not change that: $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ **diverges**.

(!) Lỗi thường gặp: Skipping the hypotheses. A justification such as "$\int_1^{\infty} f(x)\,dx$ converges, so the series converges" earns no credit for the test unless you state that f is positive, continuous, and decreasing. For the Integral Test the function must be decreasing; the test does not apply to $\sum \frac{\sin^2 n}{n^2}$, for example, since $\frac{\sin^2 x}{x^2}$ oscillates (use comparison instead).

10.5 Harmonic Series and p-Series

10.5.1 The harmonic series diverges

The **harmonic series** is $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$. Its terms tend to 0, yet it diverges. Besides the Integral Test argument above ($S_N \geq \ln(N+1)$), there is a beautiful grouping proof due to Nicole Oresme (14th century):

$$1+\frac{1}{2}+\underbrace{\left(\frac{1}{3}+\frac{1}{4}\right)}_{>\frac{1}{4}+\frac{1}{4}=\frac{1}{2}}+\underbrace{\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)}_{>4\cdot\frac{1}{8}=\frac{1}{2}}+\underbrace{\left(\frac{1}{9}+\cdots+\frac{1}{16}\right)}_{>8\cdot\frac{1}{16}=\frac{1}{2}}+\cdots$$

so $S_{2^k} > 1+\frac{k}{2}$, which is unbounded. The divergence is extremely slow: $S_{1000}\approx 7.49$, and more than 10^{43} terms are needed before the partial sums pass 100.

10.5.2 p-series

A **p-series** has the form $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$ for a constant p. For $p>0$, $f(x)=x^{-p}$ is positive, continuous and decreasing on $[1,\infty)$, and from Unit 6 (improper integrals),

$$\int_1^{\infty}\frac{dx}{x^p}=\begin{cases}\dfrac{1}{p-1}, & p>1,\\ \infty, & p\leq 1.\end{cases}$$

For $p\leq 0$ the terms do not approach 0, so the series diverges by the nth Term Test.

p-Series Test (CED 10.5)

$$\sum_{n=1}^{\infty}\frac{1}{n^p}\quad\text{converges if } p>1\quad\text{and diverges if } p\leq 1.$$

The harmonic series is the borderline case $p=1$ (diverges).

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Chuỗi điều hoà $\sum\frac{1}{n}$ phân kỳ dù $\frac{1}{n}\to 0$ — phân kỳ rất chậm (cần hơn 10^{43} số hạng để tổng vượt 100). **Chuỗi p $\sum\frac{1}{n^p}$** hội tụ khi và chỉ khi $p>1$. Mẹo nhớ: $p=1$ là "ranh giới" và thuộc phía phân kỳ. Khi dùng, phải viết rõ số mũ p bằng bao nhiêu và so với 1, ví dụ "$\sum\frac{1}{n^{3/2}}$ là chuỗi p với $p=\frac{3}{2}>1$ nên hội tụ".

Ví dụ mẫu · Recognizing p-series

Classify each series as convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n^4}}$ (b) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ (c) $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{n^{3.5}}$ (d) $\displaystyle\sum_{n=1}^{\infty}n^{-0.99}$ (e) $\displaystyle\sum_{n=1}^{\infty}\frac{5}{n^{\pi}}$

(a) $\frac{1}{n^{4/3}}$: $p=\frac{4}{3}>1$, converges. **(b)** $\frac{1}{n^{1/2}}$: $p=\frac{1}{2}\leq 1$, diverges.
(c) $\frac{n^2}{n^{3.5}}=\frac{1}{n^{1.5}}$: $p=1.5>1$, converges. **(d)** $p=0.99\leq 1$, diverges (barely — but the borderline is sharp).
(e) $5\sum\frac{1}{n^{\pi}}$ with $p=\pi>1$: converges (a constant multiple does not affect convergence).

Ví dụ mẫu · A parameter in the exponent

For which values of k does $\sum_{n=1}^{\infty} \frac{n^k}{n^{3k-1}}$ converge?

Simplify: $\frac{n^k}{n^{3k-1}} = \frac{1}{n^{2k-1}}$, a p-series with $p = 2k-1$. It converges exactly when $2k-1 > 1$, that is, **k > 1**.

Ví dụ mẫu · Integral Test with a logarithmic p

For which $p > 0$ does $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converge?

$f(x) = \frac{1}{x(\ln x)^p}$ is positive, continuous, and decreasing on $[2, \infty)$ (its denominator is a product of positive increasing functions). Substituting $u = \ln x$ turns $\int_2^{\infty} \frac{dx}{x(\ln x)^p}$ into $\int_{\ln 2}^{\infty} \frac{du}{u^p}$, which converges if and only if $p > 1$. By the Integral Test the series converges for **p > 1** and diverges for $0 < p \le 1$ (the case $p = 1$ was the divergent example above).

10.6 Comparison Tests

Most series are not exactly geometric or p-series, but many *behave like* one. The comparison tests make that intuition rigorous. Both tests require **positive terms** (at least eventually).

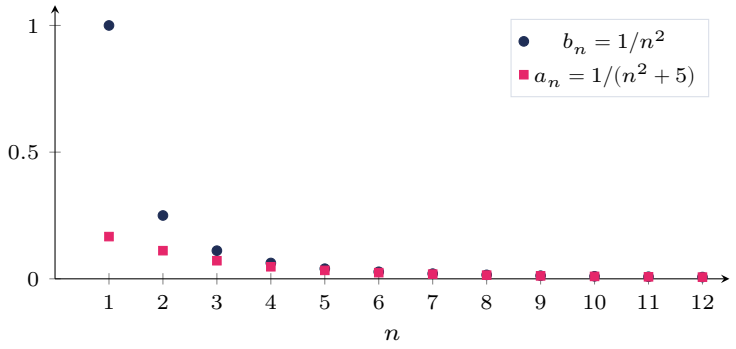
10.6.1 The Direct Comparison Test

Direct Comparison Test (CED 10.6)

Suppose $0 < a_n \le b_n$ for all n (or for all n beyond some index).

- If $\sum b_n$ **converges**, then $\sum a_n$ **converges**. (smaller than convergent ⇒ convergent)
- If $\sum a_n$ **diverges**, then $\sum b_n$ **diverges**. (larger than divergent ⇒ divergent)

The logic is the same as for the Integral Test: partial sums of a positive series increase, so they converge exactly when they are bounded. A series that sits below a convergent one is bounded by the larger sum. The other two combinations — smaller than a divergent series, or larger than a convergent one — tell us *nothing*.



Every term a_n lies below the matching term b_n, so the partial sums of $\sum a_n$ stay below $\sum b_n = \frac{\pi^2}{6}$ and must converge.

Ví dụ mẫu · Direct comparison with a p-series

Show that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+5}$ converges.

For every $n\ge 1$, $n^2+5>n^2>0$, so $0<\dfrac{1}{n^2+5}<\dfrac{1}{n^2}$. The series $\sum\frac{1}{n^2}$ is a p-series with $p=2>1$, so it converges. By the Direct Comparison Test, $\sum\frac{1}{n^2+5}$ **converges**.

Ví dụ mẫu · Direct comparison for divergence

Show that $\displaystyle\sum_{n=3}^{\infty}\frac{\ln n}{n}$ diverges (without integrating).

For $n\ge 3$, $\ln n>\ln e=1$, so $\dfrac{\ln n}{n}>\dfrac{1}{n}>0$. The harmonic series $\sum\frac{1}{n}$ diverges ($p=1$). Since our terms are *larger* than the terms of a divergent positive series, $\sum\frac{\ln n}{n}$ **diverges** by the Direct Comparison Test.

Ví dụ mẫu · Bounded numerators

Determine whether (a) $\displaystyle\sum_{n=1}^{\infty}\frac{\sin^2 n}{n^2}$ and (b) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{3^n+n}$ converge.

(a) Since $0\le\sin^2 n\le 1$, we have $0\le\dfrac{\sin^2 n}{n^2}\le\dfrac{1}{n^2}$, and $\sum\frac{1}{n^2}$ converges ($p=2>1$). By Direct Comparison the series converges. (The Integral Test cannot be used: the terms are not decreasing.)

(b) $3^n+n>3^n$, so $0<\dfrac{1}{3^n+n}<\left(\dfrac{1}{3}\right)^n$, and $\sum(\frac{1}{3})^n$ is geometric with $|r|=\frac{1}{3}<1$, hence convergent. By Direct Comparison the series converges.

10.6.2 The Limit Comparison Test

Direct comparison needs the inequality to point the right way. For $\sum\frac{1}{2^n-1}$, the natural comparison $\frac{1}{2^n-1}>\frac{1}{2^n}$ points the *wrong* way (bigger than convergent proves nothing). The Limit Comparison Test avoids inequalities altogether: it only asks whether the two terms are *proportional in the long run*.

Limit Comparison Test (CED 10.6)

Suppose $a_n>0$ and $b_n>0$ for all n (eventually), and

$$\lim_{n\to\infty}\frac{a_n}{b_n}=L,\qquad \text{where } 0<L<\infty.$$

Then $\sum a_n$ and $\sum b_n$ **both converge or both diverge**.

To choose b_n, keep only the **dominant term** of the numerator and of the denominator. For example, $\dfrac{3n^2-1}{n^4+2n}$ behaves like $\dfrac{3n^2}{n^4}=\dfrac{3}{n^2}$, so compare with $b_n=\dfrac{1}{n^2}$.

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So sánh trực tiếp: với số hạng dương, "nhỏ hơn chuỗi hội tụ thì hội tụ", "lớn hơn chuỗi phân kỳ thì phân kỳ"; hai trường hợp còn lại *không* kết luận được. **So sánh giới hạn**: nếu $\lim \frac{a_n}{b_n} = L$ với $0 < L < \infty$ thì hai chuỗi cùng tính chất. Chọn b_n bằng cách giữ lại **số hạng trội** ở tử và mẫu (thường ra chuỗi p hoặc chuỗi hình học). Khi trình bày phải ghi: $a_n, b_n > 0$; giá trị giới hạn L (hữu hạn, dương); và $\sum b_n$ hội tụ hay phân kỳ vì sao.

Ví dụ mẫu · Limit comparison, convergent

Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^3-n+4}$ converges.

For $n \geq 1$ the denominator $n^3 - n + 4 = n(n^2-1) + 4 \geq 4 > 0$, so $a_n > 0$. Dominant terms suggest $a_n \approx \frac{2n}{n^3} = \frac{2}{n^2}$; take $b_n = \frac{1}{n^2} > 0$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(2n+1)n^2}{n^3-n+4} = \lim_{n\to\infty} \frac{2n^3+n^2}{n^3-n+4} = 2.$$

Since $0 < 2 < \infty$ and $\sum \frac{1}{n^2}$ converges (p-series, $p = 2 > 1$), the series **converges** by the Limit Comparison Test.

Ví dụ mẫu · When direct comparison points the wrong way

Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ converges.

$a_n = \frac{1}{2^n-1} > 0$ for $n \geq 1$. Compare with $b_n = \frac{1}{2^n} > 0$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n}{2^n-1} = \lim_{n\to\infty} \frac{1}{1-2^{-n}} = 1.$$

Since $0 < 1 < \infty$ and $\sum(\frac{1}{2})^n$ is a convergent geometric series ($|r| = \frac{1}{2} < 1$), the series **converges** by the Limit Comparison Test.

Ví dụ mẫu · Limit comparison, divergent

Determine whether (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$ and (b) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ converge.

(a) $a_n = \frac{1}{2n-1} > 0$, $b_n = \frac{1}{n} > 0$: $\lim \dfrac{a_n}{b_n} = \lim \dfrac{n}{2n-1} = \dfrac{1}{2}$, with $0 < \frac{1}{2} < \infty$. The harmonic series diverges, so $\sum \frac{1}{2n-1}$ **diverges**.
(b) For $n \geq 1$, $0 < \frac{1}{n} \leq 1 < \pi$, so $\sin(1/n) > 0$. With $b_n = \frac{1}{n}$ and $t = \frac{1}{n} \to 0^+$,

$$\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{t\to 0^+} \frac{\sin t}{t} = 1.$$

Since $0 < 1 < \infty$ and $\sum \frac{1}{n}$ diverges, $\sum \sin(1/n)$ **diverges** by the Limit Comparison Test.

Ví dụ mẫu · Radicals

Determine whether $\displaystyle\sum_{n=1}^{\infty}\frac{n+3}{\sqrt{n^5+1}}$ converges.

Dominant terms: $\frac{n}{\sqrt{n^5}}=\frac{n}{n^{5/2}}=\frac{1}{n^{3/2}}$. With $b_n=\frac{1}{n^{3/2}}$ (both sequences positive),

$$\lim_{n\to\infty}\frac{(n+3)\,n^{3/2}}{\sqrt{n^5+1}}=\lim_{n\to\infty}\frac{(n+3)\,n^{3/2}}{n^{5/2}\sqrt{1+n^{-5}}}=\lim_{n\to\infty}\frac{1+3/n}{\sqrt{1+n^{-5}}}=1.$$

$\sum\frac{1}{n^{3/2}}$ converges ($p=\frac{3}{2}>1$), so the series **converges** by the Limit Comparison Test.

(!) Lỗi thường gặp: (1) Stating an inequality in the useless direction: "$\frac{1}{2^n-1}>\frac{1}{2^n}$ and $\sum\frac{1}{2^n}$ converges, so the series converges" is invalid. (2) Forgetting the conditions $0<L<\infty$ in the Limit Comparison Test: if $L=0$ or $L=\infty$, the basic version of the test gives no conclusion. (3) Applying either comparison test to a series with negative terms — for those, compare $\sum|a_n|$ instead (Section 10.9).

10.7 Alternating Series Test

An **alternating series** has terms that alternate in sign:

$$\sum_{n=1}^{\infty}(-1)^{n+1}b_n=b_1-b_2+b_3-b_4+\cdots\qquad\text{or}\qquad\sum_{n=1}^{\infty}(-1)^n b_n,\qquad b_n>0.$$

The alternating harmonic series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ is the model example.

Alternating Series Test (CED 10.7)

The series $\sum(-1)^{n+1}b_n$ (or $\sum(-1)^n b_n$) with $b_n>0$ **converges** if

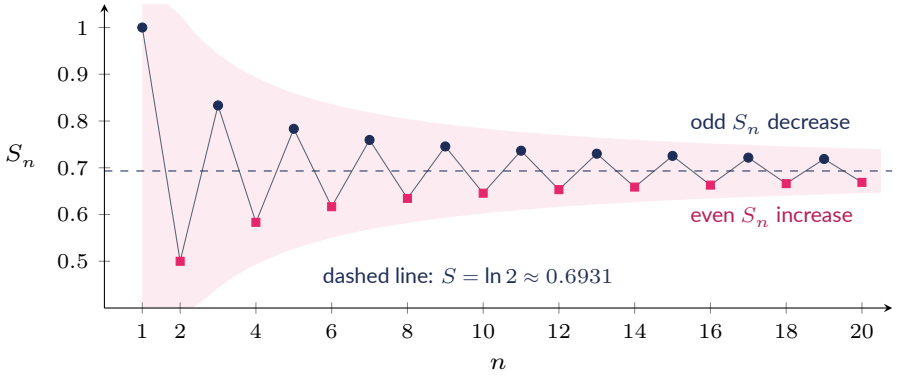
(i) the b_n are **decreasing**: $b_{n+1}\le b_n$ for all n (eventually), and

(ii) $\lim_{n\to\infty}b_n=0$.

If (ii) fails, the series diverges by the nth Term Test. If (i) fails, the test does not apply (use another method).

10.7.1 Why it works

Take b_n decreasing to 0 and look at the partial sums of $b_1-b_2+b_3-\cdots$. Every step reverses direction and is shorter than the previous step. The even partial sums $S_2,S_4,S_6,\ldots$ increase (each pair adds $b_{2k+1}-b_{2k+2}\ge 0$), the odd partial sums $S_1,S_3,S_5,\ldots$ decrease, every even sum is below every odd sum, and the gap $S_{2k+1}-S_{2k}=b_{2k+1}\to 0$. The two sides close in on a single number S.



Partial sums of $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ zig-zag around the sum $S = \ln 2$. The pink band has half-width $b_{n+1} = \frac{1}{n+1}$: the sum always lies between consecutive partial sums, and $|S - S_n| \le b_{n+1}$ — the idea behind the error bound in Section 10.10.

VN **GIẢI THÍCH TIẾNG VIỆT**

Phép thử chuỗi đan dấu: chuỗi $\sum(-1)^n b_n$ với $b_n > 0$ hội tụ nếu (i) b_n **giảm** (ít nhất từ một chỉ số nào đó) và (ii) $b_n \to 0$. Trên bài FRQ phải nêu cả ba ý: chuỗi đan dấu, $|a_n|$ giảm, $\lim |a_n| = 0$. Tổng riêng phần nhảy qua lại quanh tổng S với biên độ thu hẹp dần. Lưu ý: phép thử này chỉ chứng minh *hội tụ*; nó không cho biết hội tụ tuyệt đối hay có điều kiện.

Ví dụ mẫu · The alternating harmonic series

Show that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

The series alternates with $b_n = \frac{1}{n} > 0$. (i) $b_{n+1} = \frac{1}{n+1} < \frac{1}{n} = b_n$, so the b_n decrease. (ii) $\lim \frac{1}{n} = 0$. By the Alternating Series Test the series **converges**. (Its sum is $\ln 2$, a fact proved with Taylor series later in this unit.)

Ví dụ mẫu · Verifying "decreasing" with a derivative

Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, n}{n^2+1}$ converges.

The series alternates with $b_n = \frac{n}{n^2+1} > 0$. Let $f(x) = \frac{x}{x^2+1}$; then

$$f'(x) = \frac{(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} \le 0 \quad \text{for } x \ge 1,$$

so b_n is decreasing (strictly for $n \ge 1$: $b_1 = \frac{1}{2} > b_2 = \frac{2}{5} > \cdots$). Also $\lim \frac{n}{n^2+1} = 0$. By the Alternating Series Test the series **converges**.

Ví dụ mẫu · Eventually decreasing; and a failure

Determine whether each series converges: (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)}{n}$.

(a) For $n \ge 2$ the series alternates with $b_n = \frac{\ln n}{n} > 0$ (the $n = 1$ term is 0 and can be ignored). From the Integral Test example, $f(x) = \frac{\ln x}{x}$ has $f'(x) = \frac{1 - \ln x}{x^2} < 0$ for $x > e$, so b_n decreases for $n \ge 3$. By L'Hospital's Rule, $\lim \frac{\ln x}{x} = \lim \frac{1/x}{1} = 0$. By the Alternating Series Test the series **converges**.

(b) Here $b_n = \frac{n+1}{n} \to 1 \neq 0$. Condition (ii) fails, and in fact $\lim a_n$ does not exist, so the series **diverges** by the nth Term Test.

(!) Lỗi thường gặp: Writing only "the terms go to 0, so it converges by the Alternating Series Test." You must state that the series alternates *and* that $|a_n|$ decreases *and* that $|a_n| \to 0$. Conversely, the Alternating Series Test is not a test for divergence: if the b_n fail to decrease, the test is simply silent.

10.8 The Ratio Test

The Ratio Test asks how each term compares with the one before it. If, in the long run, each term is about L times the previous one, the series behaves like a geometric series with ratio L. It is the test of choice for terms containing **factorials** or **exponentials** c^n, and it is the main tool for power series later in the unit.

Ratio Test (CED 10.8)

For a series $\sum a_n$ with nonzero terms, let

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

- If $L < 1$, the series **converges** (in fact, converges absolutely).
- If $L > 1$ (including $L = \infty$), the series **diverges**.
- If $L = 1$, the test is **inconclusive**.

Factorial algebra. $(n+1)! = (n+1)\cdot n!$, so $\dfrac{(n+1)!}{n!} = n+1$; and $(2n+2)! = (2n+2)(2n+1)\cdot(2n)!$, so $\dfrac{(2n+2)!}{(2n)!} = (2n+2)(2n+1)$. Also $\dfrac{c^{n+1}}{c^n} = c$ and $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n = e$.

VN **GIẢI THÍCH TIẾNG VIỆT**

Phép thử tỉ số: tính $L = \lim\left|\frac{a_{n+1}}{a_n}\right|$. $L < 1$: hội tụ (tuyệt đối); $L > 1$ hoặc $L = \infty$: phân kỳ; $L = 1$: **không kết luận** — phải dùng phép thử khác. Dùng tốt nhất khi số hạng có **giai thừa** $n!$ hoặc **lũy thừa** c^n. Với các chuỗi dạng phân thức đa thức (như chuỗi p), phép thử tỉ số luôn cho $L = 1$ nên vô dụng.

Ví dụ mẫu · Exponential in the denominator

Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n}$ converges.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{3^{n+1}} \cdot \frac{3^n}{n} = \frac{n+1}{3n} \longrightarrow \frac{1}{3} = L.$$

Since $L = \frac{1}{3} < 1$, the series **converges** by the Ratio Test.

Ví dụ mẫu · Factorials

Determine whether (a) $\sum_{n=1}^{\infty} \frac{n!}{10^n}$ and (b) $\sum_{n=0}^{\infty} \frac{2^n}{n!}$ converge.

(a) $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{10^{n+1}} \cdot \frac{10^n}{n!} = \frac{n+1}{10} \to \infty$. Since $L = \infty > 1$, the series **diverges** by the Ratio Test. (The first few terms shrink, but from $n = 10$ on each term is at least as large as the previous one.)

(b) $\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \frac{2}{n+1} \to 0 = L < 1$, so the series **converges**. (Its sum is e^2, as you will see with Maclaurin series.)

Ví dụ mẫu · Double factorial growth

Determine whether $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ converges.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2} = \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{n+1}{2(2n+1)} \longrightarrow \frac{1}{4}.$$

$L = \frac{1}{4} < 1$, so the series **converges** by the Ratio Test.

Ví dụ mẫu · A limit equal to e

Determine whether $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ converges.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \frac{(n+1)^{n+1}}{(n+1)\,n^n} = \frac{(n+1)^n}{n^n} = \left(1 + \frac{1}{n}\right)^n \longrightarrow e.$$

$L = e > 1$, so the series **diverges** by the Ratio Test.

Ví dụ mẫu · When $L = 1$

Apply the Ratio Test to $\sum \frac{1}{n}$ and to $\sum \frac{1}{n^2}$.

For $\sum \frac{1}{n}$: $\frac{n}{n+1} \to 1$. For $\sum \frac{1}{n^2}$: $\frac{n^2}{(n+1)^2} \to 1$. In both cases $L = 1$, yet the first series diverges and the second converges (p-series). This is exactly why $L = 1$ gives **no conclusion**: the Ratio Test cannot distinguish them.

(!) Lỗi thường gặp: (1) When $L = 1$, do not conclude "diverges" (or "converges"); switch tests. (2) Algebra slips with factorials: $\frac{(2n+2)!}{(2n)!}$ is $(2n+2)(2n+1)$, *not* 2 or $2n+2$. (3) The Ratio Test uses $|a_{n+1}/a_n|$, not $|a_n/a_{n+1}|$; inverting the fraction turns $L = \frac{1}{3}$ into 3 and reverses the conclusion.

10.9 Absolute versus Conditional Convergence

A series with both positive and negative terms can converge for two quite different reasons: because its terms are small enough that even their absolute values add up to a finite total, or because positive and negative terms cancel delicately.

Absolute and conditional convergence (CED 10.9)

- $\sum a_n$ **converges absolutely** if $\sum |a_n|$ converges.
- $\sum a_n$ **converges conditionally** if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Absolute Convergence Theorem: if $\sum |a_n|$ converges, then $\sum a_n$ converges.

The theorem holds because $0 \le a_n + |a_n| \le 2|a_n|$; by Direct Comparison $\sum(a_n + |a_n|)$ converges, and then $\sum a_n = \sum(a_n + |a_n|) - \sum |a_n|$ is a difference of two convergent series. Its practical value: *any* positive-term test (comparison, integral, p-series, geometric, ratio) can be applied to $\sum |a_n|$, even when the original series is not alternating.

Standard procedure for a series with mixed signs.
Step 1. Test $\sum |a_n|$ with a positive-term test. If it converges $\Rightarrow$ **absolutely convergent** (done).
Step 2. If $\sum |a_n|$ diverges, test $\sum a_n$ itself (usually with the Alternating Series Test). If it converges $\Rightarrow$ **conditionally convergent**; if it diverges (usually by the nth Term Test) $\Rightarrow$ **divergent**.

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$	$p > 1$	$0 < p \le 1$	$p \le 0$
$\sum \lvert a_n \rvert = \sum \frac{1}{n^p}$	converges	diverges	diverges
$\sum a_n$ (AST / nth Term)	converges	converges	diverges
Classification	**absolutely**	**conditionally**	**divergent**

The alternating p-series shows all three behaviours as p varies.

VN **GIẢI THÍCH TIẾNG VIỆT**

Hội tụ tuyệt đối: chuỗi các trị tuyệt đối $\sum |a_n|$ hội tụ (khi đó $\sum a_n$ chắc chắn hội tụ). **Hội tụ có điều kiện**: $\sum a_n$ hội tụ nhưng $\sum |a_n|$ phân kỳ — sự hội tụ nhờ triệt tiêu giữa số hạng âm và dương. Quy trình: **(1)** xét $\sum |a_n|$ bằng phép thử cho chuỗi dương; hội tụ thì kết luận "hội tụ tuyệt đối". **(2)** Nếu $\sum |a_n|$ phân kỳ thì xét chính $\sum a_n$ (thường bằng phép thử đan dấu). Ví dụ chuẩn: $\sum \frac{(-1)^n}{n}$ hội tụ có điều kiện, $\sum \frac{(-1)^n}{n^2}$ hội tụ tuyệt đối.

Ví dụ mẫu · Absolutely, conditionally, or divergent?

Classify each series.

$$\text{(a)} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}} \qquad \text{(c)} \sum_{n=1}^{\infty} \frac{(-1)^n\, n}{n+1}$$

(a) $\sum |a_n| = \sum \frac{1}{n^2}$ is a p-series with $p = 2 > 1$: converges. The series **converges absolutely**.
(b) $\sum |a_n| = \sum \frac{1}{n^{1/2}}$ is a p-series with $p = \frac{1}{2} \le 1$: diverges. The original series alternates, $b_n = \frac{1}{\sqrt{n}}$ decreases, and $b_n \to 0$, so it converges by the AST. The series **converges conditionally**.
(c) $|a_n| = \frac{n}{n+1} \to 1 \neq 0$, so $\lim a_n \neq 0$ and the series **diverges** by the nth Term Test.

Ví dụ mẫu · A series that is not alternating

Show that $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ converges.

The signs of $\cos n$ follow no regular pattern ($\cos 1 > 0$, $\cos 2 < 0$, $\cos 3 < 0$, $\cos 4 < 0$,

$\cos 5 > 0, \dots$), so the Alternating Series Test does not apply. Instead, $0 \le \left|\frac{\cos n}{n^2}\right| \le \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges ($p = 2 > 1$). By Direct Comparison, $\sum \left|\frac{\cos n}{n^2}\right|$ converges, so the series **converges absolutely**, and therefore converges.

Ví dụ mẫu · A full AP-style justification

Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, n}{n^2+1}$ converges absolutely, converges conditionally, or diverges. Justify your answer.

Absolute values. $\sum \frac{n}{n^2+1}$ has positive terms. Compare with $b_n = \frac{1}{n} > 0$:

$$\lim_{n\to\infty} \frac{n/(n^2+1)}{1/n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1, \qquad 0 < 1 < \infty.$$

The harmonic series diverges, so $\sum \frac{n}{n^2+1}$ diverges by the Limit Comparison Test. The series does *not* converge absolutely.

Original series. It alternates; $b_n = \frac{n}{n^2+1}$ is decreasing for $n \ge 1$ (shown earlier with $f'(x) = \frac{1-x^2}{(x^2+1)^2} \le 0$) and $\lim b_n = 0$. By the Alternating Series Test, the series converges.

Conclusion. The series **converges conditionally**.

Ví dụ mẫu · Ratio Test gives absolute convergence

Classify $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$.

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} = \frac{3}{n+1} \to 0 < 1$. By the Ratio Test, $\sum |a_n|$ converges, so the series **converges absolutely**.

Why the distinction matters. The terms of an absolutely convergent series may be rearranged in any order without changing the sum. A conditionally convergent series is fragile: Riemann's Rearrangement Theorem states that its terms can be reordered to add up to *any* prescribed real number, or to diverge. For example, $1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \cdots$ (one positive term, then two negative) uses exactly the terms of the alternating harmonic series, yet sums to $\frac{1}{2}\ln 2$ instead of $\ln 2$.

10.10 Alternating Series Error Bound

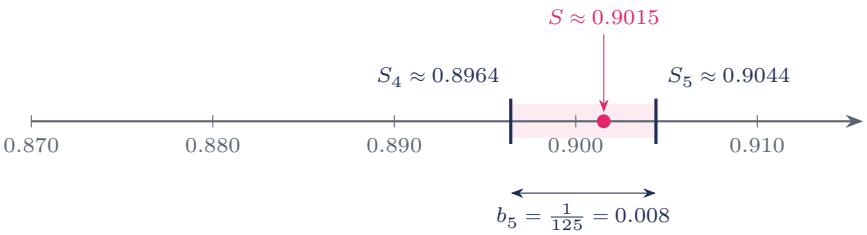
Most convergent series cannot be summed exactly, so we approximate the sum S by a partial sum S_n. The **error** (or **remainder**) is $R_n = S - S_n$. For alternating series that satisfy the AST, the picture of shrinking zig-zags gives a very simple bound.

Alternating Series Error Bound (CED 10.10)

If $\sum(-1)^{n+1}b_n$ satisfies the hypotheses of the Alternating Series Test (alternating, b_n decreasing, $b_n \to 0$) and has sum S, then

$$|S - S_n| \;\le\; b_{n+1},$$

the absolute value of the **first omitted term**. Moreover, S lies between S_n and S_{n+1}, and the error $S - S_n$ has the **same sign as the first omitted term**.



For $\sum \frac{(-1)^{n+1}}{n^3}$ the true sum S is trapped between S_4 and S_5, two consecutive partial sums whose distance is the first omitted term b_5.

VN **GIẢI THÍCH TIẾNG VIỆT**

Cận sai số chuỗi đan dấu: nếu chuỗi thoả phép thử đan dấu thì khi lấy tổng riêng phần S_n làm giá trị xấp xỉ, sai số $|S - S_n|$ **không vượt quá trị tuyệt đối của số hạng đầu tiên bị bỏ đi**, b_{n+1}. Ngoài ra S nằm giữa S_n và S_{n+1}: nếu số hạng bị bỏ đầu tiên dương thì S_n là xấp xỉ *thiếu* (underestimate), nếu âm thì S_n là xấp xỉ *thừa* (overestimate). Trên đề AP câu này thường đi kèm đa thức Taylor của một hàm số tại một điểm làm cho chuỗi đan dấu.

Ví dụ mẫu · Bounding the error of S_4

Approximate $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ by S_4, bound the error, and decide whether S_4 is too large or too small.

$S_4 = 1 - \frac{1}{8} + \frac{1}{27} - \frac{1}{64} = \frac{1549}{1728} \approx 0.8964$. The series alternates, $b_n = \frac{1}{n^3}$ decreases, and $b_n \to 0$, so the Alternating Series Error Bound applies:

$$|S - S_4| \le b_5 = \frac{1}{125} = 0.008.$$

The first omitted term, $+\frac{1}{125}$, is positive, so $S - S_4 > 0$: S_4 is an **underestimate**. Hence $0.8964 < S < 0.8964 + 0.008 = 0.9044$. (In fact $S \approx 0.9015$.)

Ví dụ mẫu · How many terms are needed?

How many terms of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ guarantee, by the Alternating Series Error Bound, that the partial sum is within 0.001 of the sum?

The series satisfies the AST ($b_n = \frac{1}{n^2}$ decreasing to 0). Using N terms, $|S - S_N| \le b_{N+1} = \frac{1}{(N+1)^2}$. We need

$$\frac{1}{(N+1)^2} < 0.001 \iff (N+1)^2 > 1000 \iff N + 1 > 31.62\ldots$$

so $N + 1 \ge 32$, i.e. **N = 31** terms suffice. (With $N = 30$ the bound is $\frac{1}{961} \approx 0.00104$, which does not guarantee the accuracy.)

Ví dụ mẫu · Overestimate or underestimate?

Let $S=\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{n\,2^n}$. (a) Find S_3. (b) Use the Alternating Series Error Bound to show $|S-S_3|<\frac{1}{60}$. (c) Is S_3 an overestimate or an underestimate?

(a) $S_3=\frac{1}{2}-\frac{1}{8}+\frac{1}{24}=\frac{12-3+1}{24}=\frac{5}{12}\approx 0.4167$.
(b) The series alternates, $b_n=\frac{1}{n2^n}$ is decreasing (both n and 2^n increase) and $b_n\to 0$. So $|S-S_3|\le b_4=\frac{1}{4\cdot 16}=\frac{1}{64}<\frac{1}{60}$.
(c) The first omitted term is $-\frac{1}{64}<0$, so $S<S_3$: S_3 is an **overestimate**. (The exact sum is $\ln\frac{3}{2}\approx 0.4055$, and indeed $S_3-S\approx 0.0112<0.0156$.)

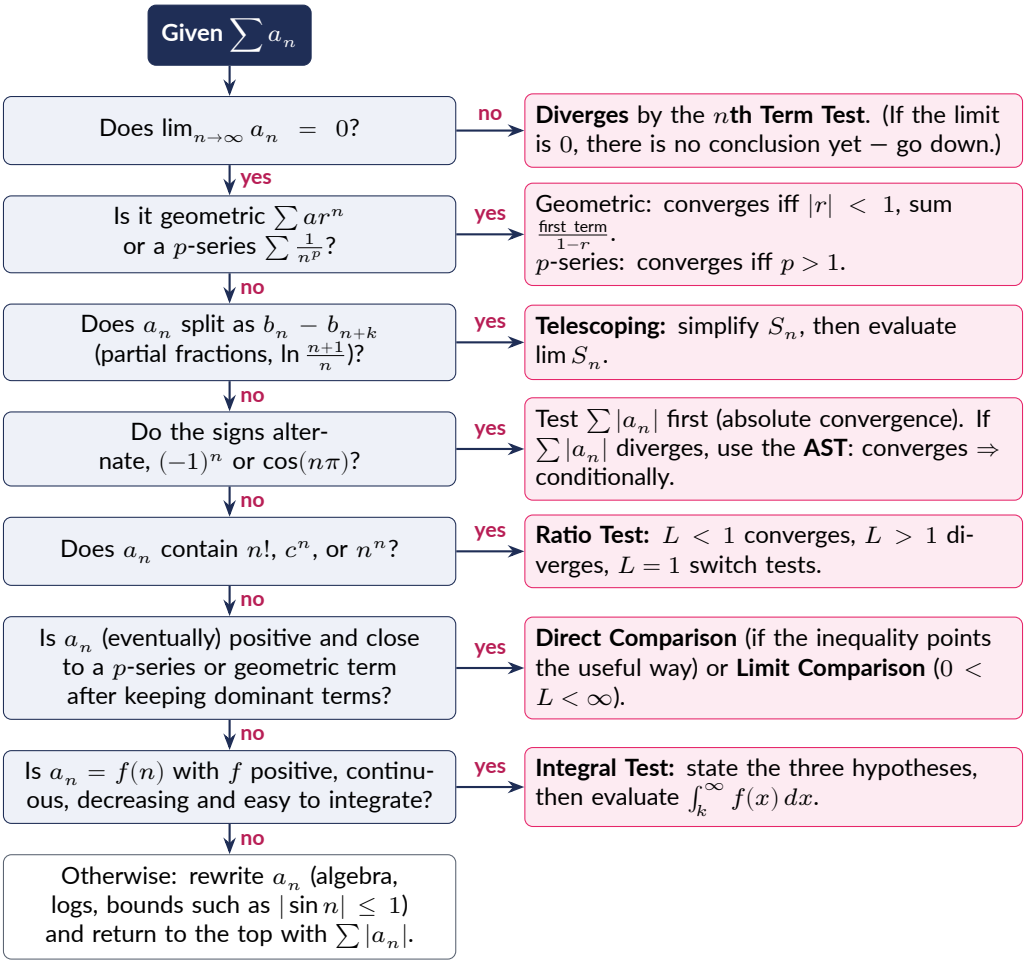
(!) Lỗi thường gặp: The bound uses the first *omitted* term b_{n+1}, not the last term used, b_n. Also, the bound is only valid when the AST hypotheses hold — state them before quoting $|S-S_n|\le b_{n+1}$. For a series that is not alternating (or whose b_n do not decrease), this error bound cannot be used; the Lagrange error bound in the next part of the unit handles Taylor polynomials in general.

10.11 Choosing a Convergence Test

On the exam, no one tells you which test to use. The skill is to *read the shape of the term* a_n and match it to a test whose hypotheses you can verify quickly. The decision chart below is a reliable order of questions; the table after it lists exactly what must be written down for each test to earn the justification point.

VN **GIẢI THÍCH TIẾNG VIỆT**

Chọn phép thử theo "hình dạng" của a_n: (1) luôn thử nhanh $\lim a_n$ — khác 0 là phân kỳ ngay; (2) nhận dạng chuỗi hình học, chuỗi p, telescoping; (3) chuỗi đan dấu: xét $\sum|a_n|$ trước, rồi mới dùng phép thử đan dấu; (4) có giai thừa hoặc lũy thừa c^n — dùng phép thử tỉ số; (5) phân thức, căn thức — so sánh (trực tiếp hoặc giới hạn) với chuỗi p; (6) hàm dễ lấy tích phân như $\frac{1}{n\ln n}$, ne^{-n^2} — dùng phép thử tích phân. Mỗi lần kết luận, ghi rõ *tên phép thử* và *các điều kiện đã kiểm tra*.



A decision chart for $\sum a_n$. The order matters: the nth Term Test costs one limit and can end the problem immediately; geometric, p-series and telescoping series are recognised by sight; alternating series need two checks (absolute, then conditional).

Test	Hypotheses you must state	Conclusion	Typical use / warning
nth Term	compute $\lim a_n$	$\lim a_n \neq 0 \Rightarrow$ diverges	never proves convergence
Geometric	$\sum ar^n$, identify r	$\lvert r\rvert < 1$: sum $\frac{a_{\text{first}}}{1-r}$; $\lvert r\rvert \geq 1$: diverges	check the first term when n starts at $1, 2, \ldots$
p-series	$\sum \frac{1}{n^p}$, identify p	$p > 1$ conv.; $p \leq 1$ div.	harmonic $p = 1$ diverges
Telescoping	$a_n = b_n - b_{n+k}$	$\sum a_n = \lim S_n$	k terms survive at each end
Integral	f positive, continuous, decreasing on $[k, \infty)$, $a_n = f(n)$	series and $\int_k^\infty f$ both conv. or both div.	integral value $\neq$ series sum
Direct Comparison	$0 < a_n \leq b_n$ (eventually)	$\sum b_n$ conv. $\Rightarrow \sum a_n$ conv.; $\sum a_n$ div. $\Rightarrow \sum b_n$ div.	direction of inequality is everything
Limit Comparison	$a_n, b_n > 0$; $\lim \frac{a_n}{b_n} = L$, $0 < L < \infty$	both conv. or both div.	choose b_n from dominant terms
Alternating Series	alternates; $\lvert a_n\rvert$ decreasing; $\lim \lvert a_n\rvert = 0$	converges	does not show absolute convergence
Ratio	$L = \lim \left\lvert \frac{a_{n+1}}{a_n} \right\rvert$	$L < 1$ conv. (abs.); $L > 1$ div.; $L = 1$?	factorials, exponentials, power series
AS Error Bound	AST hypotheses	$\lvert S - S_n\rvert \leq \lvert a_{n+1}\rvert$	sign of error = sign of a_{n+1}

Summary of the convergence tests of Topics 10.2–10.10 and the statements an AP rubric expects in a justification.

Ví dụ mẫu · Eight series, eight decisions

Determine whether each series converges or diverges. Name the test and verify its conditions.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{5n^2-n}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^2\,3^n}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$ (d) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

(e) $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n}$ (f) $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{n^2}$ (g) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right)$ (h) $\displaystyle\sum_{n=0}^{\infty} \frac{3^n + 2^n}{6^n}$

(a) $\lim \frac{n^2+1}{5n^2-n} = \frac{1}{5} \neq 0$: **diverges** by the nth Term Test.
(b) Positive terms with $\frac{2^n}{n^2 3^n} \leq \left(\frac{2}{3}\right)^n$ for $n \geq 1$; the geometric series with $r = \frac{2}{3}$ converges, so the series **converges** by Direct Comparison. (The Ratio Test also works: $L = \frac{2}{3} \lim \frac{n^2}{(n+1)^2} = \frac{2}{3} < 1$.)
(c) Positive terms; $b_n = \frac{1}{n}$: $\lim \frac{n}{n+\sqrt{n}} = \lim \frac{1}{1+n^{-1/2}} = 1$, $0 < 1 < \infty$; the harmonic series diverges, so the series **diverges** by Limit Comparison.
(d) $\lvert a_n\rvert = \frac{1}{\ln n} > \frac{1}{n} > 0$ for $n \geq 2$ (since $\ln n < n$), and $\sum \frac{1}{n}$ diverges, so $\sum \lvert a_n\rvert$ diverges by Direct Comparison. The series alternates, $\frac{1}{\ln n}$ decreases (since $\ln n$ increases) and $\to 0$, so it converges by the AST: **converges conditionally**.
(e) Ratio Test: $\frac{(n+1)^2 e^{-(n+1)}}{n^2 e^{-n}} = \left(\frac{n+1}{n}\right)^2 \frac{1}{e} \to \frac{1}{e} < 1$: **converges**.
(f) $0 < \arctan n < \frac{\pi}{2}$ for $n \geq 1$, so $0 < \frac{\arctan n}{n^2} < \frac{\pi/2}{n^2}$, and $\frac{\pi}{2} \sum \frac{1}{n^2}$ converges ($p = 2 > 1$): **converges** by Direct Comparison.
(g) Telescoping with a gap of 2: $S_n = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \to \frac{3}{2}$: **converges**, sum $\frac{3}{2}$.
(h) Split: $\sum \left(\frac{1}{2}\right)^n + \sum \left(\frac{1}{3}\right)^n = \frac{1}{1-1/2} + \frac{1}{1-1/3} = 2 + \frac{3}{2} = \frac{7}{2}$: **converges** (sum of two convergent geometric series).

(!) Lỗi thường gặp: Mixing up the two different "ratios." In a geometric series the ratio r is exactly the same for every pair of consecutive terms and its *sign* matters for the sum. In the Ratio Test, L is a *limit* of $|a_{n+1}/a_n|$ and only decides convergence — it never gives the sum.

10.12 Practice: Convergence of Series

Multiple-choice items are marked with answer choices (A)–(D). For free-response items, justify every conclusion by naming the test and verifying each of its conditions, as on the AP exam. A calculator is not needed for any item in this section.

Bài tập luyện tập

1. What is the value of $\displaystyle\sum_{n=1}^{\infty} 4\left(-\frac{1}{3}\right)^n$?

(A) 3

(B) -1

(C) 6

(D) The series diverges.

Lời giải chi tiết

Geometric with $r = -\frac{1}{3}$, $|r| < 1$, so it converges. The first term ($n = 1$) is $4 \cdot (-\frac{1}{3}) = -\frac{4}{3}$. Sum $= \dfrac{-4/3}{1-(-1/3)} = \dfrac{-4/3}{4/3} = -1$. **(B).**
Choice (A) uses 4 as the first term (as if n started at 0); choice (C) also ignores the negative sign of r.

Bài tập luyện tập

2. Which of the following series converges?

(A) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$

(B) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{0.9}}$

(C) $\displaystyle\sum_{n=1}^{\infty} (1.01)^n$

(D) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^{1.1}}$

Lời giải chi tiết

(A): terms $\to 1 \neq 0$, diverges (nth Term Test). (B): p-series, $p = 0.9 \leq 1$, diverges. (C): geometric with $r = 1.01 \geq 1$, diverges. (D): $3\sum \frac{1}{n^{1.1}}$ is a constant multiple of a p-series with $p = 1.1 > 1$, converges. **(D).**

Bài tập luyện tập

3. The nth partial sum of $\sum_{n=1}^{\infty} a_n$ is $S_n = \dfrac{2n-1}{n+3}$. What is a_3?

(A) $\dfrac{7}{30}$

(B) $\dfrac{5}{6}$

(C) 2

(D) $\dfrac{1}{4}$

Lời giải chi tiết

$a_3 = S_3 - S_2 = \frac{5}{6} - \frac{3}{5} = \frac{25-18}{30} = \frac{7}{30}$. **(A).**
(B) is S_3 itself, (C) is $\lim S_n$ (the sum of the series), and (D) is $S_1 = a_1$.

Bài tập luyện tập

4. Consider $\sum_{n=1}^{\infty} \frac{4}{(2n-1)(2n+1)}$.

(a) Show that $\frac{4}{(2n-1)(2n+1)} = \frac{2}{2n-1} - \frac{2}{2n+1}$.

(b) Find a closed-form expression for the partial sum S_n.

(c) Determine whether the series converges. If it converges, find its sum.

Lời giải chi tiết

(a) $\frac{2}{2n-1} - \frac{2}{2n+1} = \frac{2(2n+1)-2(2n-1)}{(2n-1)(2n+1)} = \frac{4}{(2n-1)(2n+1)}$.
(b) With $b_n = \frac{2}{2n-1}$, each term is $b_n - b_{n+1}$ (because $2(n+1)-1 = 2n+1$):

$$S_n = \left(2 - \tfrac{2}{3}\right) + \left(\tfrac{2}{3} - \tfrac{2}{5}\right) + \cdots + \left(\tfrac{2}{2n-1} - \tfrac{2}{2n+1}\right) = 2 - \frac{2}{2n+1} = \frac{4n}{2n+1}.$$

(c) $\lim_{n\to\infty} \frac{4n}{2n+1} = 2$. The series converges (telescoping) and its sum is **2**.

Bài tập luyện tập

5. The repeating decimal $0.7\overline{3} = 0.7333\ldots$ is equal to

(A) $\frac{73}{99}$ **(C)** $\frac{11}{15}$
(B) $\frac{73}{90}$ **(D)** $\frac{2}{3}$

Lời giải chi tiết

$0.7\overline{3} = \frac{7}{10} + (0.03 + 0.003 + \cdots)$. The repeating part is geometric with first term $\frac{3}{100}$ and $r = \frac{1}{10}$: $\frac{3/100}{9/10} = \frac{1}{30}$. So $0.7\overline{3} = \frac{7}{10} + \frac{1}{30} = \frac{22}{30} = \frac{11}{15}$. **(C).**
(A) $= 0.\overline{73}$ treats both digits as repeating; (B) $= 0.8\overline{1}$ forgets to subtract the non-repeating digit.

Bài tập luyện tập

6. Consider the series $\sum_{n=1}^{\infty} \left(\frac{2x-1}{3}\right)^n$.

(a) Find all values of x for which the series converges. Justify your answer.

(b) For the values of x found in part (a), express the sum of the series as a function of x.

(c) Find the value of x for which the sum of the series is 1.

Lời giải chi tiết

(a) Geometric with ratio $r = \frac{2x-1}{3}$. It converges if and only if $|r| < 1$: $\left|\frac{2x-1}{3}\right| < 1 \iff -3 < 2x-1 < 3 \iff -1 < x < 2$. (At $x = -1$, $r = -1$ and at $x = 2$, $r = 1$: the terms do not tend to 0, so the series diverges.)

(b) First term r (at $n = 1$): $\displaystyle\sum_{n=1}^{\infty} r^n = \frac{r}{1-r} = \frac{(2x-1)/3}{(4-2x)/3} = \frac{2x-1}{4-2x}$ for $-1 < x < 2$.

(c) $\dfrac{2x-1}{4-2x} = 1 \iff 2x-1 = 4-2x \iff x = \dfrac{5}{4}$, which lies in $(-1, 2)$. Check: $r = \frac{3/2}{3} = \frac{1}{2}$ and $\frac{1/2}{1-1/2} = 1$ ✓.

Bài tập luyện tập

7. Which of the following statements about a series $\sum a_n$ is always true?

(A) If $\sum a_n$ converges, then $\lim a_n = 0$.

(B) If $\lim a_n = 0$, then $\sum a_n$ converges.

(C) If $\sum a_n$ diverges, then $\lim a_n \neq 0$.

(D) If $\lim a_n = 0$, then $\sum a_n$ diverges.

Lời giải chi tiết

(A) is the theorem behind the nth Term Test: $a_n = S_n - S_{n-1} \to S - S = 0$. (B) and (C) are both false: the harmonic series has $\lim \frac{1}{n} = 0$ and diverges. (D) is false: $\sum \frac{1}{n^2}$ has terms $\to 0$ and converges. **(A)**.

Bài tập luyện tập

8. Which of the following series can be shown to diverge using the nth Term Test?

(A) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

(B) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

(C) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2}{n^2+5}$

(D) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

Lời giải chi tiết

Only (C) has terms with a nonzero limit: $\lim \frac{2n^2}{n^2+5} = 2 \neq 0$, so it diverges by the nth Term Test. In (A), (B), (D) the terms tend to 0 and the test is inconclusive (in fact (A) diverges, while (B) and (D) converge). **(C)**.

Bài tập luyện tập

9. Use the Integral Test to determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$ converges. Verify that the hypotheses of the test are satisfied.

Lời giải chi tiết

Let $f(x) = \dfrac{x}{(x^2+1)^2}$. For $x \geq 1$: $f(x) > 0$; f is continuous (denominator never 0); and

$$f'(x) = \frac{(x^2+1)^2 - x \cdot 2(x^2+1)(2x)}{(x^2+1)^4} = \frac{1-3x^2}{(x^2+1)^3} < 0 \quad \text{for } x \geq 1,$$

so f is decreasing. With $u = x^2+1$, $du = 2x\,dx$:

$$\int_1^\infty \frac{x\,dx}{(x^2+1)^2} = \lim_{b\to\infty}\left[-\frac{1}{2(x^2+1)}\right]_1^b = \lim_{b\to\infty}\left(-\frac{1}{2(b^2+1)} + \frac{1}{4}\right) = \frac{1}{4}.$$

The improper integral converges, so the series **converges** by the Integral Test.

Bài tập luyện tập

10. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{1+n^2}$ converges by the Integral Test. In your answer, show that the function you use is decreasing.

Lời giải chi tiết

Let $f(x) = \dfrac{\arctan x}{1+x^2}$. For $x \geq 1$, $\arctan x \geq \frac{\pi}{4} > 0$, so $f(x) > 0$, and f is continuous. By the quotient rule,

$$f'(x) = \frac{\frac{1}{1+x^2}(1+x^2) - \arctan x \cdot 2x}{(1+x^2)^2} = \frac{1-2x\arctan x}{(1+x^2)^2}.$$

For $x \geq 1$, $2x\arctan x \geq 2\cdot\frac{\pi}{4} = \frac{\pi}{2} > 1$, so $f'(x) < 0$ and f is decreasing. With $u = \arctan x$, $du = \frac{dx}{1+x^2}$:

$$\int_1^\infty \frac{\arctan x}{1+x^2}\,dx = \lim_{b\to\infty}\left[\tfrac{1}{2}(\arctan x)^2\right]_1^b = \frac{1}{2}\left(\frac{\pi^2}{4} - \frac{\pi^2}{16}\right) = \frac{3\pi^2}{32}.$$

The integral converges, so the series **converges** by the Integral Test.

Bài tập luyện tập

11. Let $S = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$. By comparing the series with $\displaystyle\int_1^\infty \frac{dx}{x^3}$, which of the following must be true?

(A) $S < 1$

(B) $S = \dfrac{1}{2}$

(C) $S > \dfrac{3}{2}$

(D) $1 < S < \dfrac{3}{2}$

Lời giải chi tiết

$f(x) = \frac{1}{x^3}$ is positive and decreasing, so the rectangles of height $\frac{1}{n^3}$ on $[n-1, n]$ for $n \geq 2$ lie under the curve: $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^3} < \int_1^\infty \frac{dx}{x^3} = \left[-\frac{1}{2x^2}\right]_1^\infty = \frac{1}{2}$. Hence $S = 1 + \sum_{n=2}^{\infty}\frac{1}{n^3} < \frac{3}{2}$, and

clearly $S > 1$ because the terms after the first are positive. **(D)**. (In fact $S \approx 1.202$.) Choice (B) confuses the value of the integral with the sum of the series.

Bài tập luyện tập

12. For which values of p does $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n^{2p}}$ converge?

(A) $p > 1$

(B) $p > 2$

(C) $p > \dfrac{3}{2}$

(D) $p \geq 2$

Lời giải chi tiết

$\dfrac{n^3}{n^{2p}} = \dfrac{1}{n^{2p-3}}$, a p-series with exponent $2p-3$. It converges iff $2p-3>1$, i.e. $p>2$. At $p=2$ the series is harmonic and diverges, which rules out (D). **(B)**.

Bài tập luyện tập

13. Which of the following correctly establishes that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n+5}$ converges?

(A) $0 < \frac{1}{2^n+5} < \frac{1}{2^n}$ for all $n \geq 1$, and $\sum \frac{1}{2^n}$ converges.

(B) $0 < \frac{1}{2^n+5} < \frac{1}{n}$ for all $n \geq 1$, and $\sum \frac{1}{n}$ diverges.

(C) $\frac{1}{2^n+5} > \frac{1}{3^n}$ for all $n \geq 3$, and $\sum \frac{1}{3^n}$ converges.

(D) $\lim_{n\to\infty} \frac{1}{2^n+5} = 0$.

Lời giải chi tiết

Direct Comparison requires the given series to be *smaller* than a *convergent* positive series; (A) does exactly that, with a geometric series of ratio $\frac{1}{2}$. (B) and (C) have the inequality in a useless direction (smaller than divergent / larger than convergent prove nothing), and (D) is the nth Term Test, which cannot prove convergence. **(A)**.

Bài tập luyện tập

14. Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2-1}{n^4+2n}$ converges or diverges. Justify your answer.

Lời giải chi tiết

For $n \geq 1$, $3n^2-1>0$ and $n^4+2n>0$, so $a_n>0$. Dominant terms give $a_n \approx \frac{3n^2}{n^4} = \frac{3}{n^2}$; take $b_n = \frac{1}{n^2} > 0$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(3n^2-1)n^2}{n^4+2n} = \lim_{n\to\infty} \frac{3n^4-n^2}{n^4+2n} = 3.$$

Since $0 < 3 < \infty$ and $\sum \frac{1}{n^2}$ converges (p-series with $p=2>1$), the series **converges** by the Limit Comparison Test.

Bài tập luyện tập

15. Determine whether $\sum_{n=1}^{\infty}\left(1-\cos\frac{1}{n}\right)$ converges or diverges. Justify your answer.

Lời giải chi tiết

For $n \geq 1$, $0 < \frac{1}{n} \leq 1 < 2\pi$, so $\cos\frac{1}{n} < 1$ and $a_n = 1-\cos\frac{1}{n} > 0$. Compare with $b_n = \frac{1}{n^2} > 0$. Put $t = \frac{1}{n} \to 0^+$ and use L'Hospital's Rule twice:

$$\lim_{n\to\infty}\frac{1-\cos(1/n)}{1/n^2} = \lim_{t\to 0^+}\frac{1-\cos t}{t^2} = \lim_{t\to 0^+}\frac{\sin t}{2t} = \lim_{t\to 0^+}\frac{\cos t}{2} = \frac{1}{2}.$$

Since $0 < \frac{1}{2} < \infty$ and $\sum\frac{1}{n^2}$ converges ($p = 2 > 1$), the series **converges** by the Limit Comparison Test.

Bài tập luyện tập

16. Determine whether $\sum_{n=1}^{\infty}\frac{2+\sin n}{\sqrt{n}}$ converges or diverges. Justify your answer.

Lời giải chi tiết

Since $-1 \leq \sin n \leq 1$, we have $2+\sin n \geq 1$, so $\frac{2+\sin n}{\sqrt{n}} \geq \frac{1}{\sqrt{n}} > 0$ for all $n \geq 1$. The series $\sum\frac{1}{n^{1/2}}$ is a p-series with $p = \frac{1}{2} \leq 1$, so it diverges. Our series has larger positive terms, so it **diverges** by the Direct Comparison Test. (Neither the Integral Test nor the Limit Comparison Test is convenient here, because $\frac{2+\sin n}{\sqrt{n}}$ is not monotone and $\frac{a_n}{b_n} = 2+\sin n$ has no limit.)

Bài tập luyện tập

17. Consider the series $\sum_{n=1}^{\infty}\frac{(-1)^n\sqrt{n}}{n+4}$.

(a) Show that the series converges.

(b) Does the series converge absolutely or conditionally? Justify your answer.

Lời giải chi tiết

(a) The series alternates with $b_n = \frac{\sqrt{n}}{n+4} > 0$. Let $f(x) = \frac{\sqrt{x}}{x+4}$:

$$f'(x) = \frac{\frac{1}{2\sqrt{x}}(x+4)-\sqrt{x}}{(x+4)^2} = \frac{(x+4)-2x}{2\sqrt{x}\,(x+4)^2} = \frac{4-x}{2\sqrt{x}\,(x+4)^2} < 0 \quad \text{for } x > 4,$$

so b_n is decreasing for $n \geq 4$. Also $\lim_{n\to\infty}\frac{\sqrt{n}}{n+4} = \lim_{n\to\infty}\frac{1/\sqrt{n}}{1+4/n} = 0$. By the Alternating Series Test, the series converges.

(b) $\sum|a_n| = \sum\frac{\sqrt{n}}{n+4}$ has positive terms. With $b_n = \frac{1}{\sqrt{n}}$: $\lim_{n\to\infty}\frac{\sqrt{n}/(n+4)}{1/\sqrt{n}} = \lim_{n\to\infty}\frac{n}{n+4} = 1$, and $0 < 1 < \infty$. Since $\sum\frac{1}{\sqrt{n}}$ diverges ($p = \frac{1}{2} \leq 1$), $\sum|a_n|$ diverges by the Limit Comparison Test. The series converges but not absolutely: it **converges conditionally**.

Bài tập luyện tập

18. When the Ratio Test is applied to $\sum_{n=1}^{\infty} \frac{5^n}{n \cdot 4^n}$, which of the following is the result?

(A) $L = \frac{4}{5}$; the series converges.

(B) $L = 1$; the test is inconclusive.

(C) $L = \frac{5}{4}$; the series diverges.

(D) $L = 5$; the series diverges.

Lời giải chi tiết

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{5^{n+1}}{(n+1)4^{n+1}} \cdot \frac{n \cdot 4^n}{5^n} = \frac{5}{4} \cdot \frac{n}{n+1} \to \frac{5}{4} > 1$, so the series diverges. **(C)**. Choice (A) comes from inverting the ratio.

Bài tập luyện tập

19. Determine whether $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2\, 5^n}$ converges or diverges. Justify your answer.

Lời giải chi tiết

All terms are positive. Ratio Test:

$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)!}{((n+1)!)^2\, 5^{n+1}} \cdot \frac{(n!)^2\, 5^n}{(2n)!} = \frac{(2n+2)(2n+1)}{(n+1)^2 \cdot 5} = \frac{2(2n+1)}{5(n+1)} \longrightarrow \frac{4}{5}.$$

Since $L = \frac{4}{5} < 1$, the series **converges** by the Ratio Test.

Bài tập luyện tập

20. For which of the following series is the Ratio Test inconclusive?

(A) $\sum_{n=1}^{\infty} \frac{3^n}{n!}$

(B) $\sum_{n=1}^{\infty} \frac{n+1}{n^3}$

(C) $\sum_{n=1}^{\infty} \frac{n}{2^n}$

(D) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

Lời giải chi tiết

(A): $L = \lim \frac{3}{n+1} = 0$. (B): $\frac{n+2}{(n+1)^3} \cdot \frac{n^3}{n+1} \to 1$: inconclusive. (C): $L = \lim \frac{n+1}{2n} = \frac{1}{2}$. (D): $\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \left(\frac{n}{n+1}\right)^n \to \frac{1}{e}$. **(B)**. (Series (B) converges by Limit Comparison with $\sum \frac{1}{n^2}$.)

Bài tập luyện tập

21. The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$

(A) converges absolutely

(B) diverges by the nth Term Test

(C) diverges because it is a p-series with $p < 1$

(D) converges conditionally

Lời giải chi tiết

$\sum |a_n| = \sum \frac{1}{n^{1/3}}$ is a p-series with $p = \frac{1}{3} \le 1$: diverges, so the convergence is not absolute. The series itself alternates, $\frac{1}{n^{1/3}}$ decreases and tends to 0: converges by the AST. **(D)**. Choice (C) wrongly applies the p-series test to an alternating series.

Bài tập luyện tập

22. Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{2^n}$ converges absolutely, converges conditionally, or diverges. Justify your answer.

Lời giải chi tiết

Apply the Ratio Test to the absolute values:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \frac{1}{2}\left(\frac{n+1}{n}\right)^2 \longrightarrow \frac{1}{2} < 1.$$

So $\sum |a_n|$ converges, and the series **converges absolutely** (hence converges).

Bài tập luyện tập

23. The series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^{3/2}}$

(A) converges absolutely

(B) converges conditionally

(C) diverges

(D) cannot be classified, because $\cos(n\pi)$ is not of the form $(-1)^n$

Lời giải chi tiết

$\cos(n\pi) = (-1)^n$, so $\left|\frac{\cos(n\pi)}{n^{3/2}}\right| = \frac{1}{n^{3/2}}$, a p-series with $p = \frac{3}{2} > 1$, which converges. The series **converges absolutely**. **(A)**.

Bài tập luyện tập

24. Let $S = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \cdot 3^n}$.

(a) Find the partial sum S_3 as a fraction.

(b) Use the Alternating Series Error Bound to find an upper bound for $|S - S_3|$. Verify that the error bound applies.

(c) Is S_3 greater than or less than S? Explain.

Lời giải chi tiết

(a) $S_3 = \frac{1}{3} - \frac{1}{18} + \frac{1}{81} = \frac{54-9+2}{162} = \frac{47}{162} \approx 0.2901$.
(b) The series alternates; $b_n = \frac{1}{n3^n}$ is decreasing because $n3^n$ is increasing; and $b_n \to 0$. So the

Alternating Series Error Bound applies: $|S - S_3| \le b_4 = \dfrac{1}{4 \cdot 81} = \dfrac{1}{324} \approx 0.00309$.

(c) The first omitted term is $\frac{(-1)^5}{4 \cdot 3^4} = -\frac{1}{324} < 0$, so $S - S_3 < 0$: S_3 is **greater than** S (an overestimate). (For reference, $S = \ln \frac{4}{3} \approx 0.2877$.)

Bài tập luyện tập

25. What is the least number of terms N for which the Alternating Series Error Bound guarantees that S_N approximates $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$ with error less than 0.0005?

(A) 4
(C) 6
(B) 5
(D) 7

Lời giải chi tiết

The AST hypotheses hold ($\frac{1}{n^4}$ decreases to 0), so $|S - S_N| \le \frac{1}{(N+1)^4}$. We need $(N+1)^4 > \frac{1}{0.0005} = 2000$, i.e. $N + 1 > 2000^{1/4} \approx 6.69$. Thus $N + 1 \ge 7$, $N = 6$. Check: $N = 5$ gives $\frac{1}{6^4} = \frac{1}{1296} \approx 0.00077$, too large; $N = 6$ gives $\frac{1}{7^4} = \frac{1}{2401} \approx 0.00042 < 0.0005$. **(C).** Choice (D) is the index of the first omitted term, not the number of terms used.

Bài tập luyện tập

26. Determine whether $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ converges absolutely, converges conditionally, or diverges. Justify your answer.

Lời giải chi tiết

Absolute values. $f(x) = \frac{1}{x \ln x}$ is positive, continuous, and decreasing on $[2, \infty)$ ($x \ln x$ is positive and increasing). With $u = \ln x$: $\displaystyle\int_2^{\infty} \frac{dx}{x \ln x} = \lim_{b \to \infty} \left[\ln(\ln x)\right]_2^b = \infty$. By the Integral Test, $\sum \frac{1}{n \ln n}$ diverges, so the convergence is not absolute.
Original series. It alternates, $b_n = \frac{1}{n \ln n}$ is decreasing (as above) and $\lim b_n = 0$. By the Alternating Series Test it converges.
Conclusion: the series **converges conditionally**.

Bài tập luyện tập

27. Which of the following series converge?

I. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ II. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$ III. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

(A) I only
(C) I and II only
(B) II only
(D) I and III only

Lời giải chi tiết

I: alternating, $\frac{1}{\ln n}$ decreasing to 0: converges (AST). II: $0 < \ln n < n$ gives $\frac{1}{\ln n} > \frac{1}{n} > 0$; the harmonic series diverges, so II diverges (Direct Comparison). III: $\ln n < \sqrt{n}$ for all $n \ge 1$, so

$0 \le \frac{\ln n}{n^2} < \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$, and $\sum \frac{1}{n^{3/2}}$ converges ($p = \frac{3}{2} > 1$): III converges (Direct Comparison). **(D)**.

Bài tập luyện tập

28. The series $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

(A) converges to 0

(B) diverges

(C) converges to $-\ln 2$

(D) converges to 1

Lời giải chi tiết

$\ln \frac{n}{n+1} = \ln n - \ln(n+1)$ telescopes: $S_n = \ln 1 - \ln(n+1) = -\ln(n+1) \to -\infty$. The series diverges. **(B)**. Choice (A) wrongly uses $\lim a_n = 0$ as if it gave the sum; (C) is only the first term.

Bài tập luyện tập

29. What is the sum of $\sum_{n=0}^{\infty} \left(\frac{e}{\pi}\right)^n$?

(A) $\dfrac{\pi}{\pi - e}$

(B) $\dfrac{e}{\pi - e}$

(C) $\dfrac{\pi - e}{\pi}$

(D) The series diverges.

Lời giải chi tiết

Geometric with $r = \frac{e}{\pi} \approx 0.865$, $|r| < 1$: converges. First term ($n = 0$) is 1, so the sum is $\dfrac{1}{1 - e/\pi} = \dfrac{\pi}{\pi - e}$. **(A)**. (B) would be correct if the series started at $n = 1$; (C) is $1 - r$, the reciprocal of the answer.

Bài tập luyện tập

30. For each series, determine whether it converges or diverges. Name the test you use and verify its conditions.

(a) $\sum_{n=1}^{\infty} \dfrac{n^3}{e^n}$ (b) $\sum_{n=1}^{\infty} \dfrac{2n}{3n-1}$ (c) $\sum_{n=1}^{\infty} \dfrac{1}{n\sqrt{n+1}}$

Lời giải chi tiết

(a) Positive terms; Ratio Test: $\dfrac{(n+1)^3}{e^{n+1}} \cdot \dfrac{e^n}{n^3} = \left(\dfrac{n+1}{n}\right)^3 \dfrac{1}{e} \to \dfrac{1}{e} < 1$: **converges**.

(b) $\lim_{n\to\infty} \dfrac{2n}{3n-1} = \dfrac{2}{3} \neq 0$: **diverges** by the nth Term Test.

(c) Positive terms; with $b_n = \frac{1}{n^{3/2}} > 0$: $\lim_{n\to\infty} \dfrac{n^{3/2}}{n\sqrt{n+1}} = \lim_{n\to\infty} \sqrt{\dfrac{n}{n+1}} = 1$, $0 < 1 < \infty$, and $\sum \frac{1}{n^{3/2}}$ converges ($p = \frac{3}{2} > 1$): **converges** by the Limit Comparison Test. (Direct Comparison also works: $\frac{1}{n\sqrt{n+1}} < \frac{1}{n\sqrt{n}}$.)

10.13 Finding Taylor Polynomial Approximations of Functions

10.13.1 From the tangent line to higher-degree approximations

In Unit 4 the **local linear approximation** of a differentiable function f near $x = a$ was the tangent line

$$L(x) = f(a) + f'(a)(x - a).$$

The tangent line is the unique polynomial of degree at most 1 that agrees with f in *value* and in *slope* at $x = a$. It is excellent very close to a, but it cannot bend: as soon as the graph of f curves away, the error grows. The natural improvement is to ask for a polynomial that also matches the **concavity** of f at a (the second derivative), then the third derivative, and so on. Each extra matching condition adds one more degree to the polynomial and lets the approximation follow the curve over a wider stretch of x-values.

Suppose we want a polynomial

$$P(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + \cdots + c_n(x - a)^n$$

with $P^{(k)}(a) = f^{(k)}(a)$ for every $k = 0, 1, \dots, n$. Differentiating P term by term and substituting $x = a$ kills every term that still contains a factor $(x - a)$:

$$P(a) = c_0, \quad P'(a) = c_1, \quad P''(a) = 2c_2, \quad P'''(a) = 3 \cdot 2\, c_3 = 3!\, c_3, \quad \dots, \quad P^{(k)}(a) = k!\, c_k.$$

Setting $P^{(k)}(a) = f^{(k)}(a)$ forces $c_k = \dfrac{f^{(k)}(a)}{k!}$. This single formula is the heart of the whole topic.

Taylor polynomial of degree n

Let f have n derivatives at $x = a$. The **nth-degree Taylor polynomial for f about $x = a$** (also called *centered at $x = a$*) is

$$\begin{aligned} P_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k \\ &= f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n. \end{aligned}$$

When the center is $a = 0$, P_n is called a **Maclaurin polynomial**. By construction P_n and f have the same value and the same first n derivatives at $x = a$. Conventions: $0! = 1$ and $f^{(0)} = f$.

Coefficient ↔ derivative. In a Taylor polynomial about $x = a$, the coefficient of $(x - a)^k$ is $c_k = \dfrac{f^{(k)}(a)}{k!}$. Read in reverse,

$$f^{(k)}(a) = k! \cdot c_k.$$

This reverse reading is tested constantly on the AP exam: given a Taylor polynomial (or series), you can recover any derivative of f at the center without differentiating anything.

VN **GIẢI THÍCH TIẾNG VIỆT**

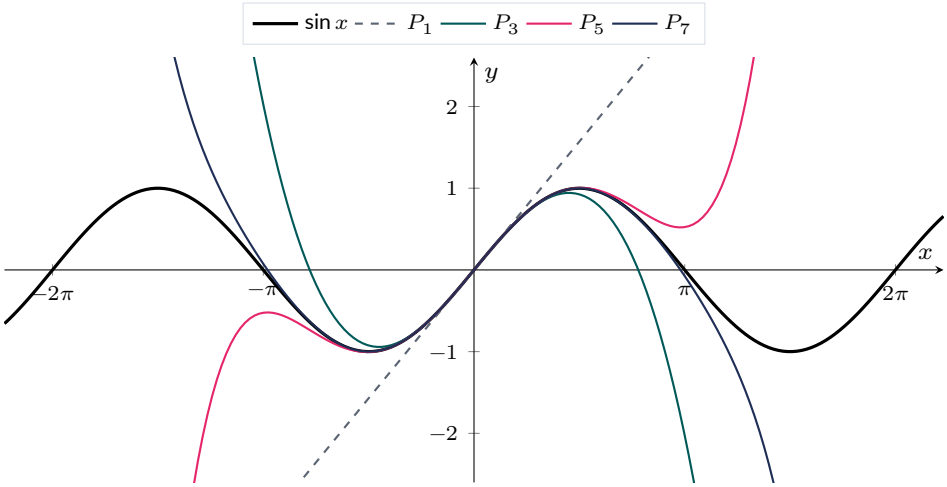
Đa thức Taylor bậc n của f tại $x = a$ là đa thức bậc $\le n$ có *cùng giá trị và cùng n đạo hàm đầu tiên* với f tại a. Hệ số của $(x-a)^k$ luôn là $\dfrac{f^{(k)}(a)}{k!}$ — đừng quên chia cho $k!$. Khi tâm $a = 0$ ta gọi là đa thức Maclaurin. Chiều ngược lại cũng rất hay ra thi: nếu biết hệ số c_k thì $f^{(k)}(a) = k!\, c_k$. Đường tiếp tuyến chính là P_1; mỗi bậc thêm vào giúp đa thức "uốn" theo đồ thị tốt hơn và xấp xỉ đúng trên một khoảng rộng hơn quanh a.

10.13.2 Taylor polynomials for the basic functions

The derivatives of $\sin x$ cycle with period four, $\sin x \to \cos x \to -\sin x \to -\cos x \to \sin x$, so at $x = 0$ the values cycle $0, 1, 0, -1$. Only odd powers survive, with alternating signs:

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}.$$

The figure below shows how each added term extends the region where the polynomial hugs the sine curve.

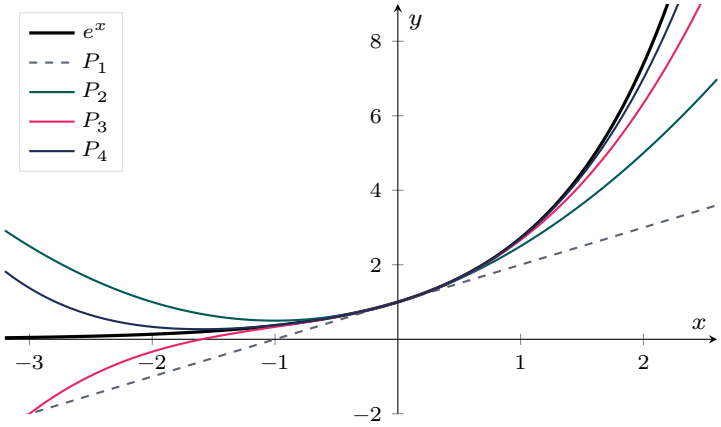


Maclaurin polynomials of degree $1, 3, 5, 7$ for $\sin x$. Near $x = 0$ all of them agree with $\sin x$; the higher the degree, the farther from the center the agreement extends (P_7 is visually indistinguishable from $\sin x$ on roughly $[-3, 3]$).

For $f(x) = e^x$ every derivative equals e^x, so $f^{(k)}(0) = 1$ for all k and

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

For $\cos x$ the values at 0 cycle $1, 0, -1, 0$, so only even powers appear: $\cos x \approx 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!}$.



Maclaurin polynomials of degree 1 through 4 for e^x. Even-degree polynomials stay above the x-axis; odd-degree ones eventually turn negative for large negative x, while e^x never does. All of them match e^x closely near $x = 0$.

Ví dụ mẫu · A Maclaurin polynomial and an approximation

Find the third-degree Maclaurin polynomial for $f(x) = e^x$ and use it to approximate $e^{0.2}$.

Every derivative of e^x is e^x, so $f(0) = f'(0) = f''(0) = f'''(0) = 1$ and

$$P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

Then

$$e^{0.2} \approx P_3(0.2) = 1 + 0.2 + \frac{0.04}{2} + \frac{0.008}{6} = 1 + 0.2 + 0.02 + 0.001\overline{3} = 1.221\overline{3}.$$

A calculator gives $e^{0.2} = 1.221402\ldots$, so the approximation is off by only about 0.00007.

Ví dụ mẫu · Taylor polynomial at a center other than 0

Find the third-degree Taylor polynomial for $f(x) = \ln x$ about $x = 1$, and use it to approximate $\ln 1.2$.

Compute derivatives and evaluate at $x = 1$:

k	0	1	2	3
$f^{(k)}(x)$	$\ln x$	x^{-1}	$-x^{-2}$	$2x^{-3}$
$f^{(k)}(1)$	0	1	-1	2
$f^{(k)}(1)/k!$	0	1	$-\frac{1}{2}$	$\frac{2}{6} = \frac{1}{3}$

Hence

$$P_3(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}.$$

With $x - 1 = 0.2$:

$$\ln 1.2 \approx 0.2 - \frac{0.04}{2} + \frac{0.008}{3} = 0.2 - 0.02 + 0.00\overline{26} = 0.18\overline{26}.$$

(The true value is $\ln 1.2 = 0.18232\ldots$.) Notice that the powers are powers of $(x-1)$, *not* of x: the polynomial is built to be accurate near the center $x = 1$.

Ví dụ mẫu · Square root near a perfect square

Find the second-degree Taylor polynomial for $f(x) = \sqrt{x}$ about $x = 4$ and use it to approximate $\sqrt{4.4}$.

$f(x) = x^{1/2}$, $f'(x) = \frac{1}{2}x^{-1/2}$, $f''(x) = -\frac{1}{4}x^{-3/2}$. At $x = 4$: $f(4) = 2$, $f'(4) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$, $f''(4) = -\frac{1}{4}\cdot\frac{1}{8} = -\frac{1}{32}$. So

$$P_2(x) = 2 + \frac{1}{4}(x-4) + \frac{-1/32}{2!}(x-4)^2 = 2 + \frac{x-4}{4} - \frac{(x-4)^2}{64}.$$

At $x = 4.4$: $P_2(4.4) = 2 + 0.1 - \dfrac{0.16}{64} = 2 + 0.1 - 0.0025 = 2.0975$. (Calculator: $\sqrt{4.4} = 2.09762\ldots$.)

(!) Lỗi thường gặp: Writing $f''(a)(x-a)^2$ instead of $\frac{f''(a)}{2!}(x-a)^2$, or $f'''(a)(x-a)^3$ instead of $\frac{f'''(a)}{3!}(x-a)^3$. The factorials are what make the derivatives of P_n match those of f. A second classic slip is expanding in powers of x when the center is $a \neq 0$: a Taylor polynomial about $x = 1$ must be written in powers of $(x-1)$.

10.13.3 Taylor polynomials from a table of derivative values

A standard free-response setup gives no formula for f at all — only a table of values of f and its derivatives at one point. The Taylor polynomial is then assembled directly from the table.

Ví dụ mẫu · FRQ-style: polynomial from a table

A function f has derivatives of all orders for all real numbers. Selected values are shown.

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$
1	2	-3	5	-12

(a) Write the third-degree Taylor polynomial for f about $x = 1$ and use it to approximate $f(1.2)$.
(b) Write the second-degree Taylor polynomial for f' about $x = 1$.
(c) Let $g(x) = \int_1^x f(t)\,dt$. Write the fourth-degree Taylor polynomial for g about $x = 1$.

(a) $P_3(x) = 2 - 3(x-1) + \frac{5}{2!}(x-1)^2 + \frac{-12}{3!}(x-1)^3 = 2 - 3(x-1) + \frac{5}{2}(x-1)^2 - 2(x-1)^3$.

$$f(1.2) \approx P_3(1.2) = 2 - 3(0.2) + \tfrac{5}{2}(0.04) - 2(0.008) = 2 - 0.6 + 0.1 - 0.016 = 1.484.$$

(b) The derivatives of f' at 1 are $f'(1) = -3$, $f''(1) = 5$, $f'''(1) = -12$, so the second-degree polynomial for f' is

$$-3 + 5(x-1) + \frac{-12}{2!}(x-1)^2 = -3 + 5(x-1) - 6(x-1)^2.$$

This is exactly $\frac{d}{dx}P_3(x)$: differentiating a Taylor polynomial term by term gives the Taylor polynomial of the derivative (one degree lower).

(c) $g(1) = 0$ and $g'(x) = f(x)$ by the Fundamental Theorem of Calculus, so the derivatives of g at 1 are $0,\ f(1),\ f'(1),\ f''(1),\ f'''(1)$. Equivalently, integrate P_3 term by term from 1 to x:

$$g(x) \approx 2(x-1) - \frac{3}{2}(x-1)^2 + \frac{5}{6}(x-1)^3 - \frac{1}{2}(x-1)^4.$$

VN GIẢI THÍCH TIẾNG VIỆT

Dạng bảng giá trị đạo hàm là dạng FRQ quen thuộc nhất: chỉ cần lắp $f(a), f'(a), f''(a), \ldots$ vào công thức, nhớ chia $k!$. Đạo hàm từng hạng của $P_n(x)$ cho đa thức Taylor của f' (bậc giảm 1); lấy nguyên hàm từng hạng từ a đến x cho đa thức Taylor của $g(x) = \int_a^x f(t)\,dt$ (bậc tăng 1, hằng số tự do bằng $g(a) = 0$).

10.13.4 Reading information about f from its Taylor polynomial

Because $f^{(k)}(a) = k!\,c_k$, a Taylor polynomial carries the local shape of f at its center: the constant term is $f(a)$, the linear coefficient is the slope, and the sign of the quadratic coefficient gives the

concavity.

Ví dụ mẫu · Second Derivative Test from a Taylor polynomial

The third-degree Taylor polynomial for a function f about $x = 2$ is $P_3(x) = 5 - 3(x-2)^2 + (x-2)^3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.

The coefficient of $(x-2)$ is 0, so $f'(2) = 1! \cdot 0 = 0$: $x = 2$ is a critical point. The coefficient of $(x-2)^2$ is -3, so $f''(2) = 2! \cdot (-3) = -6 < 0$. By the Second Derivative Test, f has a **relative maximum** at $x = 2$. (Also $f(2) = 5$ and $f'''(2) = 3! \cdot 1 = 6$, though these are not needed.)

Ví dụ mẫu · Computing the derivatives directly

Find the third-degree Maclaurin polynomial for $f(x) = \dfrac{1}{1+2x}$.

$f(x) = (1+2x)^{-1}$, $f'(x) = -2(1+2x)^{-2}$, $f''(x) = 8(1+2x)^{-3}$, $f'''(x) = -48(1+2x)^{-4}$. At 0: $1, -2, 8, -48$. Dividing by $0!, 1!, 2!, 3!$:

$$P_3(x) = 1 - 2x + \frac{8}{2}x^2 - \frac{48}{6}x^3 = 1 - 2x + 4x^2 - 8x^3.$$

Section 10.17 will produce the same answer instantly from the geometric series with ratio $-2x$.

10.14 Lagrange Error Bound

An approximation is only useful if we know how far off it might be. The difference between the function and its Taylor polynomial is the **remainder**

$$R_n(x) = f(x) - P_n(x).$$

Taylor's Theorem gives a precise formula for $R_n(x)$ involving the $(n+1)$st derivative of f at some unknown point between a and x; the AP course uses the resulting *bound*.

Lagrange Error Bound

Suppose f has $n+1$ derivatives on an interval containing a and x, and let P_n be the nth-degree Taylor polynomial for f about $x = a$. If M is a number with

$$\left|f^{(n+1)}(z)\right| \le M \quad \text{for every } z \text{ between } a \text{ and } x,$$

then

$$\left|f(x) - P_n(x)\right| \le \frac{M}{(n+1)!}\,|x-a|^{n+1}.$$

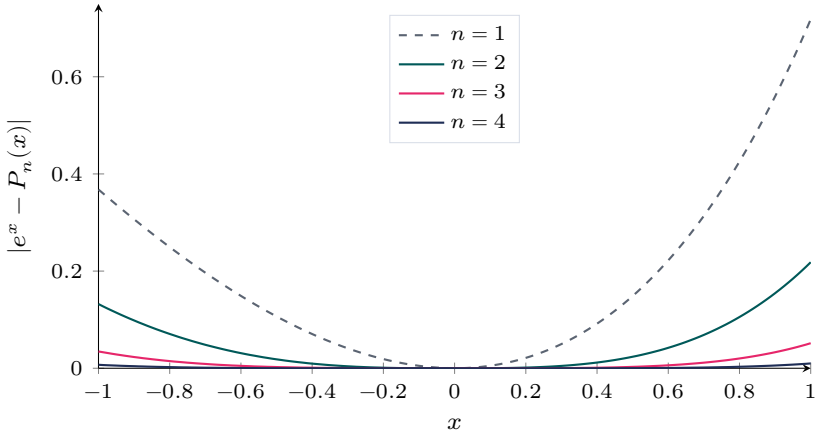
The bound looks exactly like "the next term" of the Taylor polynomial, except that the unknown derivative value is replaced by its largest possible size M.

VN **GIẢI THÍCH TIẾNG VIỆT**

Chặn sai số Lagrange: sai số khi thay $f(x)$ bằng $P_n(x)$ không vượt quá $\dfrac{M}{(n+1)!}|x-a|^{n+1}$, trong đó M là *chặn trên của* $|f^{(n+1)}|$ *trên toàn đoạn nối* a *và* x (không phải chỉ tại a). Hình dạng giống "hạng tử kế tiếp" — bậc $n+1$ — nhưng đạo hàm được thay bằng giá trị lớn nhất có thể M. Khi viết lời giải FRQ phải nêu rõ M lấy ở đâu và vì sao nó là chặn trên.

Three practical remarks:

- M need not be the *exact* maximum; any valid upper bound works (a larger M simply gives a weaker, but still correct, bound). On FRQs, M is often supplied: "$|f^{(4)}(x)| \le 15$ for $1 \le x \le 1.5$".
- The error shrinks rapidly as $|x-a|$ shrinks (a high power) and as n grows (the factorial).
- A bound $|f(x) - P_n(x)| \le E$ means $P_n(x) - E \le f(x) \le P_n(x) + E$, which is how FRQs ask you to decide whether $f(x)$ could equal a given value.



The actual error $|e^x - P_n(x)|$ for Maclaurin polynomials of e^x on $[-1, 1]$. Every curve touches 0 at the center $x = 0$ and grows like $|x|^{n+1}$ away from it; each extra degree flattens the error dramatically.

Ví dụ mẫu · Bounding the error for $e^{0.2}$

Use the Lagrange error bound to show that the approximation $e^{0.2} \approx P_3(0.2) = 1.221\overline{3}$ from the earlier example has error less than 0.0002.

Here $n = 3$, $a = 0$, $x = 0.2$, so we need a bound on $|f^{(4)}(z)| = e^z$ for $0 \le z \le 0.2$. Because e^z is increasing, $e^z \le e^{0.2} < e^1 < 3$; even more simply $e^{0.2} < 2$. Take $M = 2$:

$$\left|e^{0.2} - P_3(0.2)\right| \le \frac{2}{4!}(0.2)^4 = \frac{2(0.0016)}{24} = 0.000\overline{13} < 0.0002.$$

(The actual error is about 0.000069, comfortably inside the bound.)

Ví dụ mẫu · FRQ-style: can $f(1.2)$ equal 1.49?

For the function f of the table example (center $x = 1$, $P_3(1.2) = 1.484$), it is known that $\left|f^{(4)}(x)\right| \le 30$ for $1 \le x \le 1.2$. Use the Lagrange error bound to show that $f(1.2) \ne 1.49$.

With $n = 3$ and $M = 30$:

$$\left|f(1.2) - P_3(1.2)\right| \le \frac{30}{4!}(1.2-1)^4 = \frac{30(0.0016)}{24} = 0.002.$$

Therefore $1.484 - 0.002 \le f(1.2) \le 1.484 + 0.002$, that is, $1.482 \le f(1.2) \le 1.486$. Since $1.49 > 1.486$, $f(1.2)$ cannot equal 1.49.

Ví dụ mẫu · Choosing the degree

What is the smallest degree n for which the Maclaurin polynomial P_n for e^x is guaranteed by the Lagrange error bound to approximate $e = e^1$ with error less than 0.001?

For $0 \le z \le 1$, $|f^{(n+1)}(z)| = e^z \le e < 3$, so take $M = 3$:

$$|e - P_n(1)| \le \frac{3}{(n+1)!}(1)^{n+1} = \frac{3}{(n+1)!}.$$

We need $\frac{3}{(n+1)!} < 0.001$, i.e. $(n+1)! > 3000$. Since $6! = 720$ and $7! = 5040$, the smallest choice is $n+1 = 7$, so **n = 6**. Check: $P_6(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} = 2.71805\ldots$, and $e = 2.71828\ldots$.

10.14.1 Lagrange bound versus the alternating series error bound

When the Taylor polynomial is evaluated at a point where the series terms **alternate in sign and decrease in absolute value toward** 0, the Alternating Series Error Bound (Topic 10.10) also applies: the error is less than the absolute value of the first omitted term, and its sign matches that term. Both tools are legitimate; which one is easier depends on the problem.

	Lagrange error bound	**Alternating series error bound**
When it applies	Any Taylor polynomial, any x (given a bound M on $\lvert f^{(n+1)}\rvert$)	Only when the terms at that x alternate, decrease in size, and tend to 0
Bound	$\frac{M}{(n+1)!}\lvert x-a\rvert^{n+1}$	$\lvert$first omitted term$\rvert$
Needs	A bound M on the next derivative between a and x	Only the next nonzero term of the series
Extra information	None about the sign of the error	Sign of the error = sign of first omitted term (over/underestimate)

Ví dụ mẫu · Two bounds for $\sin 0.5$

Use $P_3(x) = x - \frac{x^3}{6}$ to approximate $\sin 0.5$, and bound the error (a) with the Lagrange error bound and (b) with the alternating series error bound.

$\sin 0.5 \approx 0.5 - \frac{0.125}{6} = 0.479\overline{16}$.

(a) Every derivative of $\sin x$ is $\pm\sin x$ or $\pm\cos x$, so $M = 1$ always works. With $n = 3$: $|\text{error}| \le \frac{1}{4!}(0.5)^4 = \frac{0.0625}{24} \approx 0.0026$. *Sharper:* the x^4 coefficient of the sine polynomial is 0, so $P_3 = P_4$, and we may use $n = 4$: $|\text{error}| \le \frac{1}{5!}(0.5)^5 = \frac{0.03125}{120} \approx 0.00026$.

(b) The Maclaurin series of $\sin x$ at $x = 0.5$ is $0.5 - \frac{0.5^3}{3!} + \frac{0.5^5}{5!} - \cdots$, which alternates with terms decreasing to 0. The first omitted term is $\frac{0.5^5}{5!} \approx 0.00026$, so the error is less than 0.00026, and since that term is positive the approximation $0.479\overline{16}$ is an **underestimate**. Indeed $\sin 0.5 = 0.479425\ldots$.

Ví dụ mẫu · Over- or underestimate?

For the approximation $\ln 1.2 \approx 0.18\overline{26}$ obtained from P_3 about $x = 1$, find an error bound and decide whether the approximation is too large or too small.

The Taylor series of $\ln x$ about 1 is $(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$. At $x = 1.2$ it becomes

$0.2 - \frac{0.2^2}{2} + \frac{0.2^3}{3} - \frac{0.2^4}{4} + \cdots$, an alternating series whose terms $\frac{0.2^k}{k}$ decrease to 0. The first omitted term is $-\dfrac{0.2^4}{4} = -0.0004$. Hence

$$\left|\ln 1.2 - 0.182\overline{6}\right| < 0.0004,$$

and because the first omitted term is *negative*, the partial sum $0.182\overline{6}$ is an **overestimate**. (Lagrange gives the same number here: $f^{(4)}(x) = -6x^{-4}$, so $|f^{(4)}| \le 6$ on $[1, 1.2]$ and the bound is $\frac{6}{24}(0.2)^4 = 0.0004$.)

(!) Lỗi thường gặp: Taking M as $|f^{(n+1)}(a)|$, the value at the center only. The Lagrange bound needs the largest size of $f^{(n+1)}$ over the *whole* interval between a and x. A second error: using the nth derivative instead of the $(n+1)$st, or dividing by $n!$ instead of $(n+1)!$. Finally, the alternating series bound may only be quoted after checking that the terms really alternate and decrease in absolute value.

10.15 Radius and Interval of Convergence of Power Series

10.15.1 Power series

A Taylor polynomial has finitely many terms. Letting the number of terms go to infinity produces a **power series**.

Power series centered at $x = a$

A **power series centered at** $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots,$$

where the coefficients c_n are constants. For each fixed value of x it is an ordinary numerical series, which may converge or diverge. The set of x-values where it converges is its **interval of convergence**, and on that set the series defines a function $f(x) = \sum c_n (x-a)^n$.

A power series always converges at its center (every term after c_0 is 0). Remarkably, the set where it converges is always an *interval* centered at a, and only three things can happen.

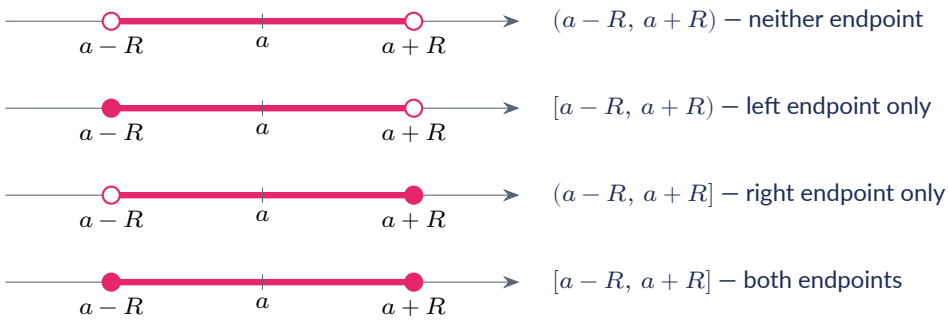
For a power series $\sum c_n (x-a)^n$, exactly one of the following holds:

(i) the series converges only at $x = a$ (**radius of convergence** $R = 0$);

(ii) the series converges for every real x ($R = \infty$, interval $(-\infty, \infty)$);

(iii) there is a number $R > 0$ such that the series converges **absolutely** for $|x-a| < R$ and diverges for $|x-a| > R$.

In case (iii), the two **endpoints** $x = a - R$ and $x = a + R$ must be tested separately: the series may converge at both, at one, or at neither. The ratio test *never* decides the endpoints.



The four possible intervals of convergence when $0 < R < \infty$. The open interval $|x-a| < R$ always belongs to the interval of convergence; a filled dot marks an endpoint where the series converges, an open dot one where it diverges.

10.15.2 Finding the radius with the ratio test

For a power series, apply the ratio test to the *absolute values* of the terms $a_n = c_n(x-a)^n$, treating x as a fixed number:

$$L(x) = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

The series converges absolutely where $L(x) < 1$ and diverges where $L(x) > 1$. Typically $L(x) = k|x-a|$ for a constant k, so the condition $k|x-a| < 1$ gives $R = 1/k$. If $L(x) = 0$ for every x, then $R = \infty$; if $L(x) = \infty$ for every $x \neq a$, then $R = 0$.

Five-step procedure for the interval of convergence

1. Write a_n (including the factor $(x-a)^n$) and form $\left|\frac{a_{n+1}}{a_n}\right|$; simplify.
2. Take the limit as $n \to \infty$; call it $L(x)$.
3. Solve $L(x) < 1$ to get the open interval and the radius R.
4. Substitute each endpoint into the *original* series and test the resulting numerical series with a test from Topics 10.3–10.9 (nth-term, p-series, comparison, alternating series, ...).
5. State the interval with the correct brackets.

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Bán kính hội tụ R tìm bằng phép thử tỉ số: tính lim $\left|\frac{a_{n+1}}{a_n}\right|$ (giữ cả x trong a_n), giải bất phương trình "< 1". Phép thử tỉ số cho giới hạn bằng 1 tại hai đầu mút nên **luôn phải thay từng đầu mút vào chuỗi ban đầu** rồi xét riêng bằng các phép thử đã học (chuỗi p, chuỗi đan dấu, so sánh, phép thử số hạng thứ n). Bỏ quên đầu mút là lỗi mất điểm phổ biến nhất ở câu hỏi khoảng hội tụ. Ba khả năng: $R = 0$ (chỉ hội tụ tại tâm), $R = \infty$ (hội tụ mọi x), hoặc $0 < R < \infty$ (cần xét đầu mút).

Ví dụ mẫu · One endpoint: $\sum x^n/n$

Find the interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$.

Ratio test. $\left|\frac{x^{n+1}/(n+1)}{x^n/n}\right| = \frac{n}{n+1}|x| \to |x|$. The series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$; $R = 1$.
$x = 1$: $\sum \frac{1}{n}$ is the harmonic series ($p = 1$), which **diverges**.
$x = -1$: $\sum \frac{(-1)^n}{n}$ is the alternating harmonic series; $\frac{1}{n}$ decreases to 0, so it **converges** by the

Alternating Series Test.
Interval of convergence: $[-1, 1)$.

Ví dụ mẫu · Both endpoints: $\sum (x-2)^n/(n^2 3^n)$

Find the radius and interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2\, 3^n}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-2|^{n+1}}{(n+1)^2\, 3^{n+1}} \cdot \frac{n^2\, 3^n}{|x-2|^n} = \frac{|x-2|}{3}\left(\frac{n}{n+1}\right)^2 \to \frac{|x-2|}{3}.$$

Convergence for $\frac{|x-2|}{3} < 1$, i.e. $|x-2| < 3$, so $R = 3$ and the open interval is $-1 < x < 5$.
$x = 5$: $\sum \dfrac{3^n}{n^2 3^n} = \sum \dfrac{1}{n^2}$, a p-series with $p = 2 > 1$: **converges**.
$x = -1$: $\sum \dfrac{(-3)^n}{n^2 3^n} = \sum \dfrac{(-1)^n}{n^2}$, which converges absolutely (its absolute values form the convergent p-series above): **converges**.
Interval of convergence: $[-1, 5]$.

Ví dụ mẫu · Neither endpoint: $\sum n(x+1)^n/4^n$

Find the interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{n(x+1)^n}{4^n}$.

$\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{n+1}{n} \cdot \dfrac{|x+1|}{4} \to \dfrac{|x+1|}{4}$, so $|x+1| < 4$: $R = 4$, open interval $-5 < x < 3$.
$x = 3$: $\sum \dfrac{n \cdot 4^n}{4^n} = \sum n$; terms $\to \infty \neq 0$, **diverges** by the nth-term test.
$x = -5$: $\sum \dfrac{n(-4)^n}{4^n} = \sum (-1)^n n$; $|a_n| = n$ does not approach 0, **diverges** by the nth-term test.
Interval of convergence: $(-5, 3)$.

Ví dụ mẫu · Right endpoint only

Find the interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x-3)^n}{n\, 2^n}$.

$\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{n}{n+1} \cdot \dfrac{|x-3|}{2} \to \dfrac{|x-3|}{2}$; $R = 2$, open interval $1 < x < 5$.
$x = 5$: $\sum \dfrac{(-1)^n 2^n}{n 2^n} = \sum \dfrac{(-1)^n}{n}$: **converges** (Alternating Series Test).
$x = 1$: $\sum \dfrac{(-1)^n (-2)^n}{n 2^n} = \sum \dfrac{(-1)^n (-1)^n}{n} = \sum \dfrac{1}{n}$: **diverges** (harmonic).
Interval of convergence: $(1, 5]$. Notice how the factor $(-1)^n$ moves the "good" endpoint from the left side (as in $\sum x^n/n$) to the right side.

Ví dụ mẫu · $R = \infty$ and $R = 0$

Find the radius of convergence of (a) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ and (b) $\displaystyle\sum_{n=0}^{\infty} n!\, x^n$.

(a) $\left|\dfrac{x^{n+1}/(n+1)!}{x^n/n!}\right| = \dfrac{|x|}{n+1} \to 0 < 1$ for *every* x. The series converges for all real x: $R = \infty$, interval $(-\infty, \infty)$. (There are no endpoints to check.)
(b) $\left|\dfrac{(n+1)!\,x^{n+1}}{n!\,x^n}\right| = (n+1)|x| \to \infty$ for every $x \neq 0$. The series converges only at its center $x = 0$: $R = 0$.

Ví dụ mẫu · A coefficient inside the power

Find the interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-1)^n}{\sqrt{n}}$.

$\left|\dfrac{a_{n+1}}{a_n}\right| = \sqrt{\dfrac{n}{n+1}}\,|2x-1| \to |2x-1|$. Solve $|2x-1| < 1$: $-1 < 2x-1 < 1$, so $0 < x < 1$. Since $|2x-1| = 2\left|x - \frac{1}{2}\right|$, the center is $\frac{1}{2}$ and the radius is $R = \frac{1}{2}$ (*not* 1).
$x = 1$: $\sum \dfrac{1}{\sqrt{n}}$ is a p-series with $p = \frac{1}{2} \le 1$: **diverges**.
$x = 0$: $\sum \dfrac{(-1)^n}{\sqrt{n}}$; $\frac{1}{\sqrt{n}}$ decreases to 0: **converges** (conditionally) by the Alternating Series Test.
Interval of convergence: $[0, 1)$.

Ví dụ mẫu · Only even powers

Find the interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{9^n}$.

$\left|\dfrac{x^{2n+2}/9^{n+1}}{x^{2n}/9^n}\right| = \dfrac{x^2}{9}$. Convergence requires $\dfrac{x^2}{9} < 1$, i.e. $|x| < 3$; $R = 3$.
$x = \pm 3$: $x^{2n} = 9^n$, so the series becomes $\sum 1$, which **diverges** (terms do not approach 0).
Interval of convergence: $(-3, 3)$. (This is a geometric series with ratio $x^2/9$, which explains the result directly.)

(!) Lỗi thường gặp: (1) Declaring the interval to be $(a-R, a+R)$ without testing the endpoints. (2) Testing an endpoint with the ratio test — at an endpoint the ratio limit is exactly 1, which is inconclusive. (3) Forgetting that a coefficient inside the power, as in $(2x-1)^n$ or $(3x)^n$, changes the radius: $|kx-b| < 1$ gives $R = 1/|k|$. (4) Losing the absolute value: the inequality is $|x-a| < R$, which is two-sided.

10.15.3 Differentiation and integration do not change the radius

If $f(x) = \sum c_n(x-a)^n$ has radius of convergence $R > 0$, then on $|x-a| < R$ the function f is differentiable and

$$f'(x) = \sum_{n=1}^{\infty} n\,c_n(x-a)^{n-1}, \qquad \int_a^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1},$$

and **both new series have the same radius** R. Their behavior at the endpoints, however, can differ from that of the original series: differentiation can lose an endpoint and integration can gain one. Endpoints must be retested for each new series.

Ví dụ mẫu · Endpoints change under differentiation

Let $f(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^2}$. Find the intervals of convergence of the series for f and for f'.

f: ratio $\frac{n^2}{(n+1)^2}|x|\to|x|$, so $R=1$. At $x=1$: $\sum\frac{1}{n^2}$ converges; at $x=-1$: $\sum\frac{(-1)^n}{n^2}$ converges absolutely. Interval: $[-1,1]$.
f': $f'(x)=\sum_{n=1}^{\infty}\frac{nx^{n-1}}{n^2}=\sum_{n=1}^{\infty}\frac{x^{n-1}}{n}$, again $R=1$. At $x=1$: $\sum\frac{1}{n}$ diverges; at $x=-1$: $\sum\frac{(-1)^{n-1}}{n}$ converges. Interval: $[-1,1)$.
Differentiation kept the radius but cost the endpoint $x=1$.

10.16 Finding Taylor or Maclaurin Series for a Function

10.16.1 Taylor series

Letting the degree of the Taylor polynomial go to infinity gives the **Taylor series**.

Taylor and Maclaurin series

If f has derivatives of all orders at $x=a$, the **Taylor series for f about $x=a$** is

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\cdots.$$

When $a=0$ it is the **Maclaurin series**. The partial sums of the Taylor series are exactly the Taylor polynomials $P_n(x)$. A complete answer on the AP exam usually asks for **the first few nonzero terms and the general term.**

Uniqueness. If $f(x)=\sum c_n(x-a)^n$ for all x in an open interval around a, then necessarily $c_n=\frac{f^{(n)}(a)}{n!}$. So *any* legitimate method of producing a power series for f (substitution, multiplication, differentiation, integration, geometric series) gives *the* Taylor series, and $f^{(n)}(a)=n!\,c_n$ can be read off from it.

For the functions in this course, the Taylor series converges to $f(x)$ throughout its interval of convergence (on the interior, and at an included endpoint the sum still equals f there when f is continuous). The AP exam expects you to know four series by heart, together with their intervals of convergence.

10.16.2 The four series to memorize

(1) e^x. Since $f^{(n)}(0)=e^0=1$ for every n, the Maclaurin series is $\sum\frac{x^n}{n!}$; the ratio test gave $R=\infty$.

(2) $\sin x$. Derivatives at 0: $0,1,0,-1,0,1,\ldots$ Only odd powers survive, with alternating signs: $\sum(-1)^n\frac{x^{2n+1}}{(2n+1)!}$. Ratio: $\frac{x^2}{(2n+2)(2n+3)}\to 0$, so $R=\infty$.

(3) $\cos x$. Derivatives at 0: $1,0,-1,0,\ldots$ Only even powers: $\sum(-1)^n\frac{x^{2n}}{(2n)!}$, $R=\infty$. (Differentiating the sine series term by term gives the cosine series, as it must.)

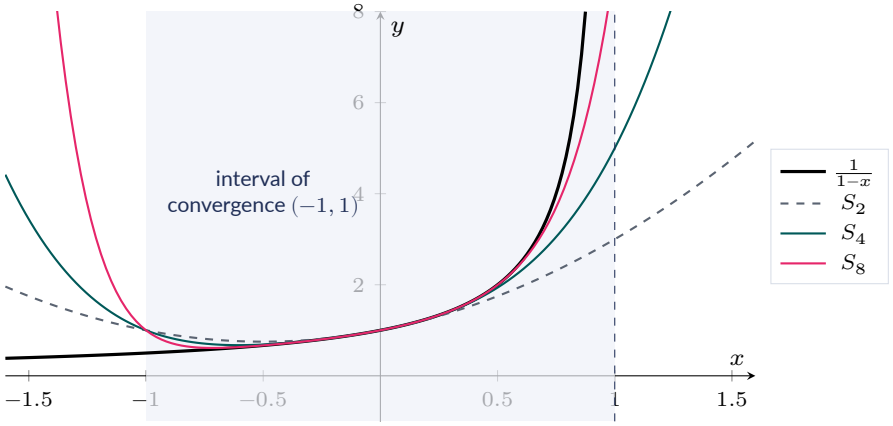
(4) $\frac{1}{1-x}$. Here $f^{(n)}(x)=\frac{n!}{(1-x)^{n+1}}$, so $f^{(n)}(0)=n!$ and $c_n=1$: the series is $\sum x^n$, the geometric series with ratio x, converging exactly when $|x|<1$.

Function	Maclaurin series	Interval of convergence
e^x	$1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots + \dfrac{x^n}{n!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \dfrac{x^n}{n!}$	$(-\infty, \infty)$
$\sin x$	$x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots + \dfrac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$	$(-\infty, \infty)$
$\cos x$	$1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots + \dfrac{(-1)^n x^{2n}}{(2n)!} + \cdots$	$(-\infty, \infty)$
$\dfrac{1}{1-x}$	$1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \displaystyle\sum_{n=0}^{\infty} x^n$	$(-1, 1)$
Derived in the next section (useful to know as well):		
$\ln(1+x)$	$x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \cdots + \dfrac{(-1)^{n+1} x^n}{n} + \cdots$	$(-1, 1]$
$\arctan x$	$x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \cdots + \dfrac{(-1)^n x^{2n+1}}{2n+1} + \cdots$	$[-1, 1]$

The Maclaurin series that must be known by heart on the AP Calculus BC exam, with their intervals of convergence.

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Bốn chuỗi **bắt buộc thuộc lòng**: e^x, $\sin x$, $\cos x$ (hội tụ với mọi x) và $\dfrac{1}{1-x} = \sum x^n$ (chỉ hội tụ khi $|x| < 1$). Mẹo nhớ: sin là hàm lẻ nên chỉ có lũy thừa lẻ, cos là hàm chẵn nên chỉ có lũy thừa chẵn; cả hai đan dấu. Chuỗi e^x giống sin cộng cos nhưng không đan dấu. Tính **duy nhất** của chuỗi lũy thừa cho phép ta lấy chuỗi bằng bất kỳ cách hợp lệ nào (thế, nhân, đạo hàm, tích phân) và vẫn được đúng chuỗi Taylor.



Partial sums $S_N = 1 + x + \cdots + x^N$ of the geometric series. Inside $(-1, 1)$ (shaded) they close in on $\frac{1}{1-x}$; outside, they separate wildly even where $\frac{1}{1-x}$ itself is perfectly well defined (for example at $x = -1.5$).

10.16.3 Taylor series by direct differentiation

When none of the known series applies directly, compute the derivatives, look for a pattern, and write the general term.

Ví dụ mẫu · Taylor series of $1/x$ about $x = 1$

Find the Taylor series for $f(x) = \dfrac{1}{x}$ about $x = 1$, including the general term, and its interval of convergence.

$f(x) = x^{-1}$, $f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$, $f'''(x) = -6x^{-4}$; in general $f^{(n)}(x) = (-1)^n n!\, x^{-(n+1)}$, so $f^{(n)}(1) = (-1)^n n!$ and

$$c_n = \frac{(-1)^n n!}{n!} = (-1)^n, \qquad \frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots + (-1)^n (x-1)^n + \cdots.$$

This is geometric with ratio $-(x-1)$, so it converges exactly when $|x-1| < 1$; at $x = 0$ and $x = 2$ the terms have absolute value 1 and the series diverges. **Interval:** $(0, 2)$.

Ví dụ mẫu · Taylor series of e^x about $x = 2$

Find the Taylor series for $f(x) = e^x$ about $x = 2$.
Every derivative is e^x, so $f^{(n)}(2) = e^2$ and

$$e^x = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n = e^2\left[1 + (x-2) + \frac{(x-2)^2}{2!} + \frac{(x-2)^3}{3!} + \cdots\right],$$

valid for all real x. (Check: $e^x = e^2 e^{x-2}$, and e^{x-2} is the e^u series with $u = x-2$.)

Ví dụ mẫu · FRQ-style: series from a derivative formula

A function f has derivatives of all orders at $x = 0$, with $f(0) = 1$ and

$$f^{(n)}(0) = \frac{(-1)^n\, n!}{3^n(n+1)} \quad \text{for } n \geq 1.$$

(a) Write the first three nonzero terms and the general term of the Maclaurin series for f.
(b) Find the interval of convergence of the series.

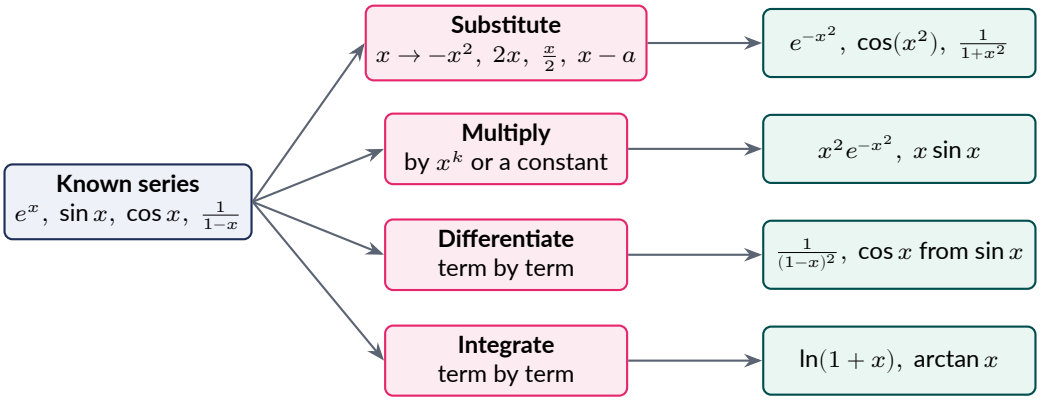
(a) $c_n = \dfrac{f^{(n)}(0)}{n!} = \dfrac{(-1)^n}{3^n(n+1)}$ (and the formula also gives $c_0 = 1 = f(0)$). So

$$f(x) = 1 - \frac{x}{6} + \frac{x^2}{27} - \cdots + \frac{(-1)^n x^n}{3^n(n+1)} + \cdots.$$

(b) $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{n+1}{n+2} \cdot \dfrac{|x|}{3} \to \dfrac{|x|}{3}$, so $R = 3$. At $x = 3$: $\sum \dfrac{(-1)^n}{n+1}$ converges by the Alternating Series Test ($\frac{1}{n+1}$ decreases to 0). At $x = -3$: $\sum \dfrac{1}{n+1}$ diverges (harmonic). **Interval:** $(-3, 3]$.

10.17 Representing Functions as Power Series

Computing $f^{(n)}(a)$ for every n is often messy or impossible to pattern. Fortunately, uniqueness of power series means that we can manufacture new series from the four known ones by algebra and calculus, and the result is automatically the Taylor series.



Four legitimate operations turn the memorized series into new ones. Substitution can change the interval of convergence; differentiation and integration keep the radius but require the endpoints to be rechecked.

10.17.1 Substitution

If $f(u) = \sum c_n u^n$ for $|u| < R$, replacing u by an expression in x gives a series for the composite function, valid wherever the substituted expression satisfies $|u| < R$.

Ví dụ mẫu · Substituting into the geometric series

Find Maclaurin series, with intervals of convergence, for (a) $\dfrac{1}{1+x^2}$ and (b) $\dfrac{1}{2-x}$.

(a) Replace u by $-x^2$ in $\frac{1}{1-u} = \sum u^n$:

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots,$$

valid when $|-x^2| < 1$, i.e. $|x| < 1$. At $x = \pm 1$ the series is $1 - 1 + 1 - \cdots$, which diverges. Interval $(-1, 1)$.

(b) First force the form $\frac{1}{1-u}$ by factoring out 2:

$$\frac{1}{2-x} = \frac{1}{2} \cdot \frac{1}{1-\frac{x}{2}} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}} = \frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \cdots,$$

valid for $\left|\frac{x}{2}\right| < 1$, i.e. $|x| < 2$; at $x = \pm 2$ the terms have size $\frac{1}{2}$, so it diverges. Interval $(-2, 2)$.

Ví dụ mẫu · Substituting into e^x and $\cos x$

Write the first four nonzero terms and the general term of the Maclaurin series for (a) e^{-x^2} and (b) $\cos(x^2)$.

(a) With $u = -x^2$ in $e^u = \sum \frac{u^n}{n!}$:

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + \frac{(-1)^n x^{2n}}{n!} + \cdots, \quad \text{all real } x.$$

(b) With $u = x^2$ in $\cos u = \sum (-1)^n \frac{u^{2n}}{(2n)!}$:

$$\cos(x^2) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots + \frac{(-1)^n x^{4n}}{(2n)!} + \cdots, \quad \text{all real } x.$$

10.17.2 Multiplying by a power of x

Multiplying a series by x^k shifts every exponent up by k and does not change where it converges.

Ví dụ mẫu · $x^2e^{-x^2}$ and a high-order derivative

(a) Find the Maclaurin series for $f(x) = x^2e^{-x^2}$. (b) Find $f^{(6)}(0)$ and $f^{(7)}(0)$.

(a) Multiply the series for e^{-x^2} by x^2:

$$x^2e^{-x^2} = x^2 - x^4 + \frac{x^6}{2!} - \frac{x^8}{3!} + \cdots + \frac{(-1)^n x^{2n+2}}{n!} + \cdots, \quad \text{all real } x.$$

(b) The coefficient of x^6 is $c_6 = \frac{1}{2!} = \frac{1}{2}$, so $f^{(6)}(0) = 6! \cdot \frac{1}{2} = 360$. There is no x^7 term, so $c_7 = 0$ and $f^{(7)}(0) = 0$. Computing six derivatives of $x^2e^{-x^2}$ by hand would be a long and error-prone exercise; the series makes it a one-line computation.

Ví dụ mẫu · $\dfrac{x}{1+4x^2}$

Find the Maclaurin series and interval of convergence for $g(x) = \dfrac{x}{1+4x^2}$.

With $u = -4x^2$: $\dfrac{1}{1+4x^2} = \sum(-4x^2)^n = \sum(-1)^n 4^n x^{2n}$, valid for $4x^2 < 1$, i.e. $|x| < \frac{1}{2}$. Multiply by x:

$$\frac{x}{1+4x^2} = x - 4x^3 + 16x^5 - \cdots + (-1)^n 4^n x^{2n+1} + \cdots, \qquad |x| < \tfrac{1}{2}.$$

At $x = \pm\frac{1}{2}$ the terms are $\pm\frac{1}{2}$ in size and do not approach 0, so the interval is $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

10.17.3 Differentiating and integrating term by term

Ví dụ mẫu · Differentiating the geometric series

Find a power series for $\dfrac{1}{(1-x)^2}$ and use it to find $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$.

Differentiate $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ term by term (valid for $|x| < 1$):

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots, \qquad |x| < 1.$$

Multiply by x: $\dfrac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n x^n$. At $x = \frac{1}{2}$ (inside the interval):

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1/2}{(1-1/2)^2} = \frac{1/2}{1/4} = 2.$$

Ví dụ mẫu · The series for $\ln(1+x)$

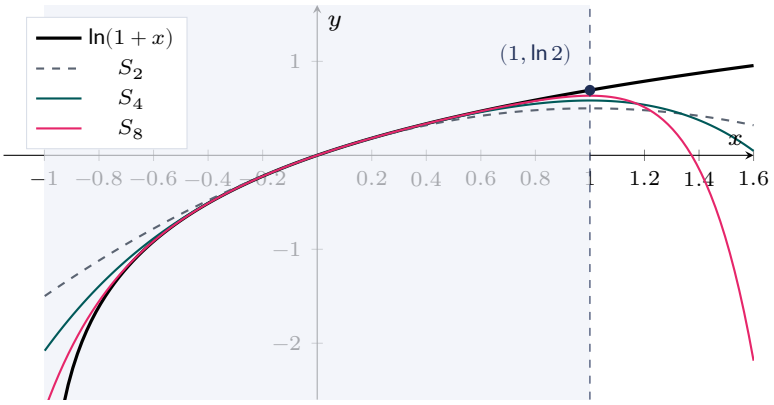
Find the Maclaurin series for $\ln(1+x)$ and its interval of convergence.

Since $\frac{d}{dx}\ln(1+x) = \frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$ for $|x| < 1$, integrate from 0 to x (and use $\ln 1 = 0$):

$$\ln(1+x) = \int_0^x \frac{dt}{1+t} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$$

The radius stays 1. $x = 1$: $1 - \frac{1}{2} + \frac{1}{3} - \cdots$ converges (alternating harmonic). $x = -1$: $-\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots\right)$ diverges. Interval: $(-1, 1]$. Integration gained the endpoint $x = 1$, where the series converges to $\ln 2$:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2.$$



Partial sums of the series for $\ln(1+x)$. On $(-1, 1]$ (shaded, right edge included) the partial sums approach $\ln(1+x)$, slowly near the endpoints; for $x > 1$ they peel away from the curve, the higher-degree ones faster, even though $\ln(1+x)$ is defined there.

Ví dụ mẫu · The series for arctan x

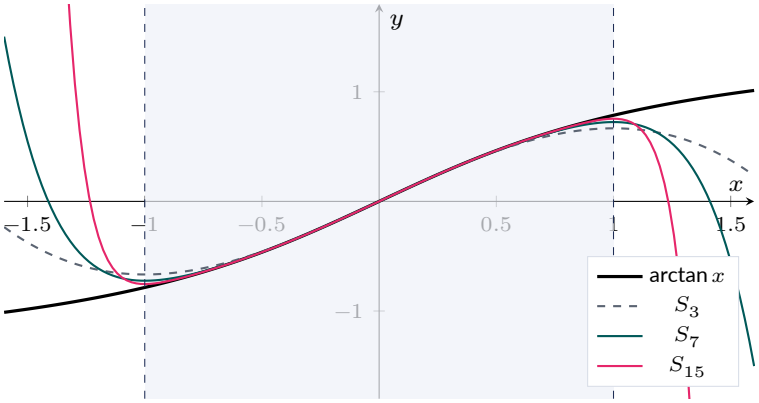
Find the Maclaurin series for $\arctan x$ and its interval of convergence.

$\frac{d}{dx}\arctan x = \frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$ for $|x| < 1$. Integrate from 0 to x (with $\arctan 0 = 0$):

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$$

$x = 1$: $1 - \frac{1}{3} + \frac{1}{5} - \cdots$ converges (Alternating Series Test). $x = -1$: the negative of the same series, also convergent. Interval: $[-1, 1]$. At $x = 1$ this yields the Leibniz formula

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \arctan 1 = \frac{\pi}{4}.$$



Partial sums S_N (through the x^N term) of the series for arctan x. Inside $[-1, 1]$ they converge to arctan x; just beyond $|x| = 1$ higher-degree partial sums peel away faster, a picture of divergence for $|x| > 1$.

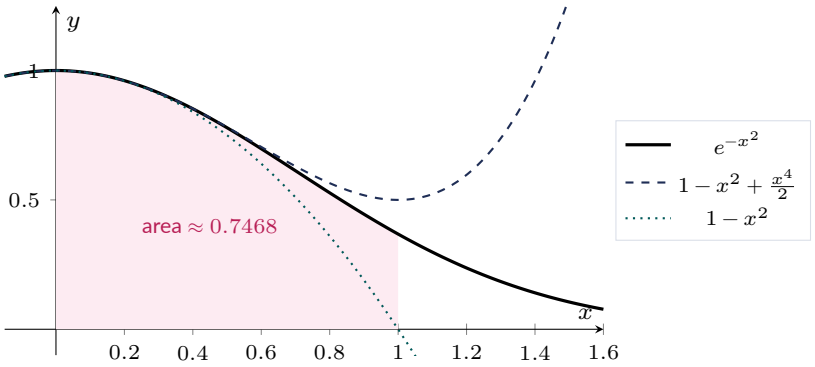
GIẢI THÍCH TIẾNG VIỆT VN

Lấy chuỗi mới từ chuỗi đã biết bằng 4 thao tác: **thế** ($x \to -x^2$, $\frac{x}{2}$, ...; khoảng hội tụ thay đổi theo), **nhân** với x^k (khoảng hội tụ giữ nguyên), **đạo hàm** và **tích phân** từng hạng (giữ bán kính, phải xét lại đầu mút). Hai chuỗi quan trọng thu được bằng tích phân: $\ln(1+x)$ hội tụ trên $(-1, 1]$ và $\arctan x$ hội tụ trên $[-1, 1]$. Khi gặp $\frac{1}{a-x}$, luôn rút a ra để được dạng $\frac{1}{1-u}$ trước khi thế.

(!) Lỗi thường gặp: Forgetting the constant of integration. When you integrate term by term to get a series for F, either integrate from the center ($\int_0^x$) or add C and determine it from a known value such as $\ln 1 = 0$ or $\arctan 0 = 0$. Also: when substituting into $\frac{1}{1-u}$, the substituted expression must appear exactly as "$1 - u$": $\frac{1}{1+x^2}$ uses $u = -x^2$, not $u = x^2$.

10.17.4 Approximating definite integrals with series

Some integrals, such as $\int e^{-x^2}\,dx$ or $\int \frac{\sin x}{x}\,dx$, have no elementary antiderivative. Integrating the Maclaurin series term by term still gives their exact value as a series, and the Alternating Series Error Bound controls the accuracy.



The shaded area $\int_0^1 e^{-x^2}\,dx$ has no elementary antiderivative, but on $[0, 1]$ the partial sums of the Maclaurin series of e^{-x^2} lie alternately below and above the curve, and integrating them gives alternating approximations of the area.

Ví dụ mẫu · $\displaystyle\int_0^1 e^{-x^2}\,dx$ to within 0.001

Use a Maclaurin series to approximate $\displaystyle\int_0^1 e^{-x^2}\,dx$ with error less than 0.001. Justify the accuracy.

Integrate the series for e^{-x^2} term by term:

$$\int_0^1 e^{-x^2}\,dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(2n+1)} = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \frac{1}{1320} + \cdots.$$

This is an alternating series whose terms $\frac{1}{n!(2n+1)}$ decrease to 0. The term $\frac{1}{1320} \approx 0.00076$ is the first one smaller than 0.001, so we stop just before it:

$$\int_0^1 e^{-x^2}\,dx \approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} \approx 0.7475,$$

with error less than $\frac{1}{1320} < 0.001$ by the Alternating Series Error Bound. (A calculator gives $0.746824\ldots$; and since the first omitted term is negative, our value is an overestimate, as observed.)

Ví dụ mẫu · The sine integral

Approximate $\displaystyle\int_0^1 \frac{\sin x}{x}\,dx$ using the first three nonzero terms of a series, and bound the error.

Dividing the sine series by x: $\dfrac{\sin x}{x} = 1 - \dfrac{x^2}{3!} + \dfrac{x^4}{5!} - \dfrac{x^6}{7!} + \cdots$ (the value 1 at $x = 0$ removes the singularity). Then

$$\int_0^1 \frac{\sin x}{x}\,dx = 1 - \frac{1}{3\cdot 3!} + \frac{1}{5\cdot 5!} - \frac{1}{7\cdot 7!} + \cdots = 1 - \frac{1}{18} + \frac{1}{600} - \frac{1}{35280} + \cdots.$$

First three terms: $1 - \frac{1}{18} + \frac{1}{600} \approx 0.94611$. The series alternates with decreasing terms, so the error is less than $\frac{1}{35280} < 0.00003$. (True value: $0.946083\ldots$)

10.17.5 Finding the sum of a series

Recognizing a numerical series as a known Maclaurin series evaluated at a specific x gives its exact sum.

Ví dụ mẫu · Recognizing known series

Find the exact sum of each series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{6^{2n+1}(2n+1)!}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\,3^n}$ (d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\,3^n}$

(a) This is $\sum \frac{x^n}{n!}$ with $x = 2$: the sum is e^2.

(b) Write the general term as $\dfrac{(-1)^n(\pi/6)^{2n+1}}{(2n+1)!}$: the sine series at $x = \frac{\pi}{6}$. Sum $= \sin\frac{\pi}{6} = \frac{1}{2}$.

(c) $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$ with $x = \frac{1}{3}$ (inside $(-1, 1]$). Sum $= \ln\frac{4}{3}$.

(d) The odd powers and the denominators $2n+1$ suggest $\arctan x$. Since $\frac{1}{3^n} = \left(\frac{1}{\sqrt{3}}\right)^{2n} = \sqrt{3}\left(\frac{1}{\sqrt{3}}\right)^{2n+1}$, the series equals $\sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n(1/\sqrt{3})^{2n+1}}{2n+1} = \sqrt{3}\arctan\frac{1}{\sqrt{3}} = \sqrt{3}\cdot\frac{\pi}{6} = \frac{\pi\sqrt{3}}{6}$.

(!) Lỗi thường gặp: Before quoting a sum, confirm that the x-value lies in the interval of convergence and that the starting index matches the memorized series. For instance, $\sum_{n=1}^{\infty}\frac{x^n}{n!} = e^x - 1$ because the $n=0$ term (1) is missing.

10.17.6 Limits with series

Replacing functions by their series turns many $\frac{0}{0}$ limits into polynomial algebra.

Ví dụ mẫu · Limits by series

Evaluate (a) $\lim_{x\to 0}\dfrac{x-\sin x}{x^3}$ and (b) $\lim_{x\to 0}\dfrac{e^{x^2}-1-x^2}{x^4}$.

(a) $x-\sin x = x-\left(x-\frac{x^3}{6}+\frac{x^5}{120}-\cdots\right) = \frac{x^3}{6}-\frac{x^5}{120}+\cdots$. Dividing by x^3: $\frac{1}{6}-\frac{x^2}{120}+\cdots\to\frac{1}{6}$.

(b) $e^{x^2} = 1+x^2+\frac{x^4}{2}+\frac{x^6}{6}+\cdots$, so $e^{x^2}-1-x^2 = \frac{x^4}{2}+\frac{x^6}{6}+\cdots$. Dividing by x^4: $\frac{1}{2}+\frac{x^2}{6}+\cdots\to\frac{1}{2}$.

(Part (a) by L'Hospital's Rule would need three applications; the series does it in one line.)

VN **GIẢI THÍCH TIẾNG VIỆT**

Ba ứng dụng của chuỗi lũy thừa hay ra thi: (1) **tính tích phân không có nguyên hàm sơ cấp** — lấy tích phân từng hạng rồi dùng chặn sai số chuỗi đan dấu để biết cần bao nhiêu hạng; (2) **tính tổng chuỗi số** — nhận ra chuỗi quen thuộc (e^x, $\sin x$, $\cos x$, $\ln(1+x)$, $\arctan x$, chuỗi hình học) tại một giá trị x cụ thể, nhớ kiểm tra x nằm trong khoảng hội tụ và chỉ số bắt đầu; (3) **tính giới hạn dạng** $\frac{0}{0}$ — thay hàm bằng chuỗi, rút gọn lũy thừa thấp nhất.

10.18 Problem Bank

The problems below cover all of Unit 10: convergence of numerical series and Taylor polynomials and power series. Multi-part free-response items follow the structure of the final question on the AP Calculus BC exam.

Bài tập luyện tập

What is the sum of the series $\displaystyle\sum_{n=1}^{\infty} 3\left(-\frac{2}{5}\right)^{n-1}$?

(A) 5

(B) $\dfrac{15}{7}$

(C) $-\dfrac{6}{7}$

(D) The series diverges.

Lời giải chi tiết

Geometric series with first term $a=3$ (the $n=1$ term) and ratio $r=-\frac{2}{5}$. Since $|r|<1$ it converges, with sum

$$\frac{a}{1-r} = \frac{3}{1+\frac{2}{5}} = \frac{3}{7/5} = \frac{15}{7}.$$

(B). Choice (A) comes from dropping the minus sign in r; choice (C) uses $-\frac{6}{5}$ as the first term (the $n = 2$ term).

Bài tập luyện tập

Which of the following series converges conditionally?

(A) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

(B) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+1}$

(C) $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n}$

(D) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

Lời giải chi tiết

(A) converges absolutely ($\sum \frac{1}{n^2}$ is a p-series, $p = 2$). (B) diverges: $\frac{n}{n+1} \to 1$, so the terms do not approach 0. (C) converges absolutely (geometric, $r = \frac{1}{3}$). (D): $\frac{1}{\sqrt{n}}$ decreases to 0, so the series converges by the Alternating Series Test, but $\sum \frac{1}{\sqrt{n}}$ is a p-series with $p = \frac{1}{2} \le 1$ and diverges. Hence (D) converges conditionally. **(D)**.

Bài tập luyện tập

Use the integral test to determine whether $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges or diverges. Verify the hypotheses of the test.

Lời giải chi tiết

Let $f(x) = \frac{1}{x(\ln x)^2}$ for $x \ge 2$. On $[2, \infty)$, f is continuous and positive (since $x > 0$ and $\ln x > 0$), and decreasing because $x(\ln x)^2$ is a product of positive increasing functions. Substitute $u = \ln x$, $du = \frac{dx}{x}$:

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{du}{u^2} = \lim_{b\to\infty} \left[-\frac{1}{u}\right]_{\ln 2}^{\ln b} = \lim_{b\to\infty} \left(\frac{1}{\ln 2} - \frac{1}{\ln b}\right) = \frac{1}{\ln 2}.$$

The improper integral converges, so the series **converges** by the integral test. (The integral's value is not the sum of the series.)

Bài tập luyện tập

Which of the following series diverge?

I. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ II. $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2}$ III. $\sum_{n=1}^{\infty} \frac{2^n}{3^n+1}$

(A) I only

(B) II only

(C) I and III only

(D) I, II, and III

Lời giải chi tiết

I: limit comparison with $\sum \frac{1}{n}$: $\dfrac{n/(n^2+1)}{1/n} = \dfrac{n^2}{n^2+1} \to 1$, a positive finite limit, and $\sum \frac{1}{n}$ diverges, so I diverges. **II:** $0 \le \frac{\sin^2 n}{n^2} \le \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges, so II converges by direct comparison. **III:** $0 < \frac{2^n}{3^n+1} < \left(\frac{2}{3}\right)^n$, a convergent geometric series, so III converges. Only I diverges. **(A)**.

Bài tập luyện tập

The sum S_4 of the first four terms is used to approximate $S = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$. Which of the following is the smallest upper bound for $|S - S_4|$ guaranteed by the alternating series error bound?

(A) $\dfrac{1}{64}$

(B) $\dfrac{1}{216}$

(C) $\dfrac{1}{125}$

(D) $\dfrac{1}{25}$

Lời giải chi tiết

The terms $\frac{1}{n^3}$ decrease to 0 and alternate in sign, so the error after S_4 is less than the absolute value of the first omitted term, $a_5 = \frac{1}{5^3} = \frac{1}{125}$. **(C)**. Choice (A) is the last *included* term; (B) skips a term.

Bài tập luyện tập

For the series $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{n \cdot 4^n}$, the ratio test gives which limit and conclusion?

(A) $\frac{4}{5}$; converges

(B) $\frac{5}{4}$; converges

(C) $\frac{5}{4}$; diverges

(D) 1; inconclusive

Lời giải chi tiết

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{5^{n+1}}{(n+1)4^{n+1}} \cdot \frac{n\,4^n}{5^n} = \frac{5}{4} \cdot \frac{n}{n+1} \to \frac{5}{4} > 1,$$

so the series diverges. **(C)**. (Indeed the terms $\frac{(5/4)^n}{n} \to \infty$.)

Bài tập luyện tập

Which statement about $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n+1}$ is true?

(A) It diverges by the nth-term test, because $\lim\limits_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \neq 0$.

(B) It converges, because $\frac{n}{2n+1} < \frac{1}{2}$ for every n.

(C) It converges to $\frac{1}{2}$.

(D) The nth-term test is inconclusive, so the ratio test must be used.

Lời giải chi tiết

$\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \neq 0$, so the series diverges by the nth-term test. **(A)**. Choice (C) confuses the limit of the *terms* with the sum of the series; choice (B) is not a valid test (bounded terms do not imply convergence); and the ratio test would give limit 1 here, which is inconclusive anyway.

Bài tập luyện tập

Determine whether each series converges absolutely, converges conditionally, or diverges. Justify your answers.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ (b) $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{2^n}$ (c) $\sum_{n=1}^{\infty} \frac{n\cos(n\pi)}{3n-1}$

Lời giải chi tiết

(a) Absolute values: $\frac{n}{n^2+1}$; limit comparison with $\frac{1}{n}$ gives ratio $\frac{n^2}{n^2+1} \to 1$, and $\sum \frac{1}{n}$ diverges, so the series does *not* converge absolutely. Alternating Series Test: let $g(x) = \frac{x}{x^2+1}$; $g'(x) = \frac{1-x^2}{(x^2+1)^2} \leq 0$ for $x \geq 1$, so the terms decrease, and $\frac{n}{n^2+1} \to 0$. The series converges. **Conditionally convergent.**

(b) Ratio test on absolute values: $\frac{(n+1)^2/2^{n+1}}{n^2/2^n} = \frac{1}{2}\left(\frac{n+1}{n}\right)^2 \to \frac{1}{2} < 1$. **Absolutely convergent.**

(c) $\cos(n\pi) = (-1)^n$, so the terms are $\frac{(-1)^n n}{3n-1}$ with $\left|\frac{(-1)^n n}{3n-1}\right| \to \frac{1}{3} \neq 0$. The terms do not approach 0: **diverges** by the nth-term test.

Bài tập luyện tập

A function f satisfies $f'''(0) = 12$. What is the coefficient of x^3 in the Maclaurin series for f?

(A) 12 **(C)** 72

(B) 4 **(D)** 2

Lời giải chi tiết

$c_3 = \frac{f'''(0)}{3!} = \frac{12}{6} = 2$. **(D)**. Choice (B) divides by 3 instead of $3!$; choice (A) forgets the factorial.

Bài tập luyện tập

The second-degree Maclaurin polynomial for $f(x) = \sqrt{1+x}$ is

(A) $1 + \frac{x}{2} - \frac{x^2}{4}$ **(C)** $1 + \frac{x}{2} + \frac{x^2}{8}$

(B) $1 + \frac{x}{2} - \frac{x^2}{8}$ **(D)** $1 - \frac{x}{2} + \frac{3x^2}{8}$

Lời giải chi tiết

$f(x)=(1+x)^{1/2}$, $f'(x)=\frac{1}{2}(1+x)^{-1/2}$, $f''(x)=-\frac{1}{4}(1+x)^{-3/2}$. At 0: $1,\ \frac{1}{2},\ -\frac{1}{4}$. So

$$P_2(x)=1+\frac{1}{2}x+\frac{-1/4}{2!}x^2=1+\frac{x}{2}-\frac{x^2}{8}.$$

(B). Choice (A) omits the $2!$; choice (D) is the polynomial for $(1+x)^{-1/2}$.

Bài tập luyện tập

A function f has $f(2)=1$, $f'(2)=4$, $f''(2)=-6$, and $f'''(2)=18$. Using the third-degree Taylor polynomial for f about $x=2$, the approximation of $f(2.1)$ is

(A) 1.373
(C) 1.370
(B) 1.358
(D) 1.433

Lời giải chi tiết

$P_3(x)=1+4(x-2)-\frac{6}{2!}(x-2)^2+\frac{18}{3!}(x-2)^3=1+4(x-2)-3(x-2)^2+3(x-2)^3$. With $x-2=0.1$:

$$P_3(2.1)=1+0.4-0.03+0.003=1.373.$$

(A). Choice (B) omits the factorials; (C) stops at degree 2; (D) mishandles the sign of the quadratic term.

Bài tập luyện tập

Let $P_4(x)$ be the fourth-degree Maclaurin polynomial for a function f, and suppose $\left|f^{(5)}(x)\right|\le 8$ for all x in $[0,1]$. Of the following, which is the smallest upper bound for $|f(1)-P_4(1)|$ given by the Lagrange error bound?

(A) $\frac{1}{3}$
(C) $\frac{1}{90}$
(B) $\frac{8}{5}$
(D) $\frac{1}{15}$

Lời giải chi tiết

With $n=4$, $M=8$, $|x-a|=1$:

$$|f(1)-P_4(1)|\le\frac{8}{5!}\cdot 1^5=\frac{8}{120}=\frac{1}{15}.$$

(D). Choice (A) divides by $4!$; choice (C) divides by $6!$.

Bài tập luyện tập

What is the interval of convergence of $\displaystyle\sum_{n=1}^{\infty}\frac{(x-3)^n}{n\,5^n}$?

(A) $(-2,8)$
(C) $[-2,8)$
(B) $[-2,8]$
(D) $(-2,8]$

Lời giải chi tiết

Ratio: $\frac{n}{n+1}\cdot\frac{|x-3|}{5}\to\frac{|x-3|}{5}<1\iff -2<x<8$. At $x=8$: $\sum\frac{5^n}{n5^n}=\sum\frac{1}{n}$ diverges (harmonic). At $x=-2$: $\sum\frac{(-5)^n}{n5^n}=\sum\frac{(-1)^n}{n}$ converges (Alternating Series Test). Interval $[-2,8)$. **(C)**.

Bài tập luyện tập

What is the radius of convergence of $\displaystyle\sum_{n=1}^{\infty}\frac{n!\,x^n}{n^n}$?

(A) 1

(B) e

(C) $\dfrac{1}{e}$

(D) ∞

Lời giải chi tiết

$$\left|\frac{a_{n+1}}{a_n}\right|=\frac{(n+1)!\,|x|^{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{n!\,|x|^n}=|x|\cdot\frac{(n+1)\,n^n}{(n+1)^{n+1}}=|x|\left(\frac{n}{n+1}\right)^n=\frac{|x|}{\left(1+\frac{1}{n}\right)^n}\to\frac{|x|}{e}.$$

Convergence for $\frac{|x|}{e}<1$, i.e. $|x|<e$. $R=e$. **(B)**.

Bài tập luyện tập

Which of the following is the Maclaurin series for $x\cos(2x)$?

(A) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\,2\,x^{2n+1}}{(2n)!}$

(B) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\,4^n\,x^{2n+1}}{(2n)!}$

(C) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\,4^n\,x^{2n}}{(2n)!}$

(D) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\,2^{2n+1}x^{2n+2}}{(2n+1)!}$

Lời giải chi tiết

$\cos u=\sum\frac{(-1)^n u^{2n}}{(2n)!}$ with $u=2x$ gives $\cos 2x=\sum\frac{(-1)^n 2^{2n}x^{2n}}{(2n)!}=\sum\frac{(-1)^n 4^n x^{2n}}{(2n)!}$. Multiplying by x raises every power by one: $x\cos 2x=\sum\frac{(-1)^n 4^n x^{2n+1}}{(2n)!}=x-2x^3+\frac{2}{3}x^5-\cdots$. **(B)**. Choice (C) forgets to multiply by x; choice (A) does not raise 2 to the power $2n$.

Bài tập luyện tập

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{\pi}{3}\right)^{2n}=$$

(A) $\dfrac{\sqrt{3}}{2}$

(B) 0

(C) $-\dfrac{1}{2}$

(D) $\dfrac{1}{2}$

Lời giải chi tiết

This is the Maclaurin series $\cos x = \sum \frac{(-1)^n x^{2n}}{(2n)!}$ evaluated at $x = \frac{\pi}{3}$, which converges for all x. Sum $= \cos\frac{\pi}{3} = \frac{1}{2}$. **(D)**. Choice (A) is $\sin\frac{\pi}{3}$.

Bài tập luyện tập

$$\lim_{x\to 0} \frac{e^{2x} - 1 - 2x}{x^2} =$$

(A) 2

(B) $\frac{1}{2}$

(C) 1

(D) 4

Lời giải chi tiết

$e^{2x} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \cdots$. Then $\frac{e^{2x} - 1 - 2x}{x^2} = 2 + \frac{4}{3}x + \cdots \to 2$. **(A)**. Choice (D) forgets the $2!$ in the denominator.

Bài tập luyện tập

Let $f(x) = x^2 \sin x$. What is the value of $f^{(5)}(0)$?

(A) $-\frac{1}{6}$

(B) 20

(C) -20

(D) -120

Lời giải chi tiết

$x^2 \sin x = x^2\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) = x^3 - \frac{x^5}{6} + \frac{x^7}{120} - \cdots$. The coefficient of x^5 is $c_5 = -\frac{1}{6}$, so $f^{(5)}(0) = 5!\, c_5 = 120 \cdot \left(-\frac{1}{6}\right) = -20$. **(C)**. Choice (A) is the coefficient itself, not the derivative.

Bài tập luyện tập

Let $F(x) = \displaystyle\int_0^x \ln(1+t)\,dt$. What is the coefficient of x^4 in the Maclaurin series for F?

(A) $\frac{1}{12}$

(B) $-\frac{1}{12}$

(C) $\frac{1}{3}$

(D) $-\frac{1}{4}$

Lời giải chi tiết

$\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots$. Integrating from 0 to x: $F(x) = \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \cdots$. The x^4 term comes from $\int_0^x \frac{t^3}{3}\,dt = \frac{x^4}{12}$. **(A)**. Choice (C) forgets to integrate; choice (D) uses the t^4 term of $\ln(1+t)$.

Bài tập luyện tập

The approximation $\cos 1 \approx 1 - \frac{1}{2} + \frac{1}{24}$ is made using the Maclaurin series for $\cos x$. Which of the following is the smallest upper bound for the error guaranteed by the alternating series error bound?

(A) $\frac{1}{24}$

(B) $\frac{1}{720}$

(C) $\frac{1}{120}$

(D) $\frac{1}{5040}$

Lời giải chi tiết

At $x = 1$ the series is $1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots$, alternating with terms decreasing to 0. The approximation uses the terms through $\frac{1}{4!}$; the first omitted term is $\frac{1}{6!} = \frac{1}{720}$. **(B).** Choice (C), $\frac{1}{5!}$, is not a term of the cosine series (only even powers appear).

Bài tập luyện tập

Which of the following gives the Maclaurin series for $\frac{1}{1+3x}$ and its interval of convergence?

(A) $\sum_{n=0}^{\infty} 3^n x^n,\ |x| < \frac{1}{3}$

(B) $\sum_{n=0}^{\infty} (-3)^n x^n,\ |x| < 3$

(C) $\sum_{n=0}^{\infty} (-1)^n 3x^n,\ |x| < 1$

(D) $\sum_{n=0}^{\infty} (-3)^n x^n,\ |x| < \frac{1}{3}$

Lời giải chi tiết

$\frac{1}{1+3x} = \frac{1}{1-(-3x)} = \sum(-3x)^n = \sum(-3)^n x^n$, valid for $|-3x| < 1$, i.e. $|x| < \frac{1}{3}$. At $x = \pm\frac{1}{3}$ the terms have absolute value 1, so the interval is open. **(D).**

Bài tập luyện tập

Which of the following is the third-degree Taylor polynomial for $f(x) = e^{2x}$ about $x = 1$?

(A) $e^2[1 + 2(x-1) + 4(x-1)^2 + 8(x-1)^3]$

(B) $e^2\left[1 + (x-1) + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6}\right]$

(C) $e^2\left[1 + 2(x-1) + 2(x-1)^2 + \frac{4}{3}(x-1)^3\right]$

(D) $1 + 2(x-1) + 2(x-1)^2 + \frac{4}{3}(x-1)^3$

Lời giải chi tiết

$f^{(k)}(x) = 2^k e^{2x}$, so $f^{(k)}(1) = 2^k e^2$: values $e^2,\ 2e^2,\ 4e^2,\ 8e^2$. Dividing by $k!$: $e^2,\ 2e^2,\ \frac{4e^2}{2} = 2e^2,\ \frac{8e^2}{6} = \frac{4}{3}e^2$. **(C).** Choice (A) omits the factorials, (B) omits the chain-rule factors 2^k, and (D) omits $f(1) = e^2$.

Bài tập luyện tập

What is the sum of $\sum_{n=1}^{\infty} \frac{n}{3^n}$?

(A) $\frac{1}{2}$ **(C)** $\frac{4}{3}$

(B) $\frac{9}{4}$ **(D)** $\frac{3}{4}$

Lời giải chi tiết

From $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$ ($|x| < 1$), multiply by x: $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$. At $x = \frac{1}{3}$:

$$\sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{1/3}{(2/3)^2} = \frac{1/3}{4/9} = \frac{3}{4}.$$

(D). Choice (B), $\frac{9}{4}$, is $\sum n\left(\frac{1}{3}\right)^{n-1}$ — forgetting to multiply by x.

Bài tập luyện tập

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\,2^n}{n\,3^n} =$$

(A) $\ln\frac{5}{3}$ **(C)** $-\ln\frac{5}{3}$

(B) $\ln 3$ **(D)** $\frac{3}{5}$

Lời giải chi tiết

$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$ for $-1 < x \le 1$. The series is this with $x = \frac{2}{3}$ (inside the interval), so the sum is $\ln\left(1+\frac{2}{3}\right) = \ln\frac{5}{3}$. **(A)**. Choice (B) is $-\ln\left(1-\frac{2}{3}\right)$, the value of the non-alternating series $\sum \frac{(2/3)^n}{n}$.

Bài tập luyện tập

Find the radius and interval of convergence of $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n^2+1}$.

Lời giải chi tiết

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n^2+1}{(n+1)^2+1}\,|3x-2| \to |3x-2|$. Convergence for $|3x-2| < 1$: $\frac{1}{3} < x < 1$. Since $|3x-2| = 3\left|x-\frac{2}{3}\right|$, the radius is $R = \frac{1}{3}$.
$x = 1$: $\sum \frac{1}{n^2+1}$; $0 < \frac{1}{n^2+1} < \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges, so it converges by direct comparison.
$x = \frac{1}{3}$: $\sum \frac{(-1)^n}{n^2+1}$ converges absolutely by the same comparison.
Interval of convergence: $\left[\frac{1}{3}, 1\right]$.

Bài tập luyện tập

Use the first two nonzero terms of a Maclaurin series to approximate $\int_0^{0.5} \sin(x^2)\,dx$, and show that the approximation is within 10^{-6} of the exact value.

Lời giải chi tiết

$\sin(x^2) = x^2 - \dfrac{x^6}{3!} + \dfrac{x^{10}}{5!} - \cdots$. Integrating term by term,

$$\int_0^{0.5} \sin(x^2)\,dx = \frac{(0.5)^3}{3} - \frac{(0.5)^7}{7\cdot 3!} + \frac{(0.5)^{11}}{11\cdot 5!} - \cdots.$$

First two terms: $\dfrac{0.125}{3} - \dfrac{0.0078125}{42} \approx 0.041667 - 0.000186 = 0.041481$. The series alternates with decreasing terms, so the error is less than the next term $\dfrac{(0.5)^{11}}{1320} \approx 3.7 \times 10^{-7} < 10^{-6}$.

Bài tập luyện tập

A function f has derivatives of all orders for all real x, with

$$f(0) = 2, \qquad f'(0) = -1, \qquad f''(0) = 4, \qquad f'''(0) = -9,$$

and $\left|f^{(4)}(x)\right| \le 15$ for $0 \le x \le 0.5$.
(a) Write the third-degree Taylor polynomial $P_3(x)$ for f about $x = 0$, and use it to approximate $f(0.5)$.
(b) Use the Lagrange error bound to show that $f(0.5) < 1.86$.
(c) Write the second-degree Taylor polynomial for f' about $x = 0$, and use it to approximate $f'(0.5)$.
(d) Let $g(x) = \displaystyle\int_0^x f(t)\,dt$. Write the fourth-degree Taylor polynomial for g about $x = 0$.

Lời giải chi tiết

(a) $P_3(x) = 2 - x + \dfrac{4}{2!}x^2 + \dfrac{-9}{3!}x^3 = 2 - x + 2x^2 - \dfrac{3}{2}x^3$.

$$f(0.5) \approx P_3(0.5) = 2 - 0.5 + 2(0.25) - 1.5(0.125) = 2 - 0.5 + 0.5 - 0.1875 = 1.8125.$$

(b) With $n = 3$, $M = 15$:

$$|f(0.5) - P_3(0.5)| \le \frac{15}{4!}(0.5)^4 = \frac{15(0.0625)}{24} = 0.0390625.$$

So $f(0.5) \le 1.8125 + 0.0390625 = 1.8515625 < 1.86$.

(c) $f'(0) = -1$, $f''(0) = 4$, $f'''(0) = -9$, so the polynomial for f' is $-1 + 4x - \dfrac{9}{2}x^2$ (which is $P_3'(x)$). Then $f'(0.5) \approx -1 + 2 - \dfrac{9}{2}(0.25) = -1 + 2 - 1.125 = -0.125$.

(d) $g(0) = 0$ and $g' = f$; integrate P_3 term by term from 0 to x:

$$g(x) \approx 2x - \frac{x^2}{2} + \frac{2x^3}{3} - \frac{3x^4}{8}.$$

Bài tập luyện tập

The third-degree Taylor polynomial for a function f about $x = 3$ is

$$P_3(x) = 4 - 2(x-3) + 5(x-3)^2 - \tfrac{1}{2}(x-3)^3.$$

(a) Find $f(3)$, $f'(3)$, $f''(3)$, and $f'''(3)$.
(b) Does f have a relative extremum at $x = 3$? Is the graph of f concave up or concave down at $x = 3$? Justify.
(c) Write the second-degree Taylor polynomial for f' about $x = 3$ and use it to approximate $f'(3.2)$.
(d) It is known that $\left|f^{(4)}(x)\right| \le 6$ for $3 \le x \le 3.2$. Approximate $f(3.2)$ using P_3, and give an interval guaranteed to contain $f(3.2)$.

Lời giải chi tiết

(a) Using $f^{(k)}(3) = k!\,c_k$: $f(3) = 4$, $f'(3) = -2$, $f''(3) = 2! \cdot 5 = 10$, $f'''(3) = 3! \cdot \left(-\frac{1}{2}\right) = -3$.
(b) $f'(3) = -2 \neq 0$, so $x = 3$ is not a critical point and f has **no** relative extremum there. Since $f''(3) = 10 > 0$, the graph of f is concave up at $x = 3$.
(c) $-2 + 10(x-3) + \dfrac{-3}{2!}(x-3)^2 = -2 + 10(x-3) - \dfrac{3}{2}(x-3)^2$. At $x = 3.2$: $-2 + 2 - 1.5(0.04) = -0.06$.
(d) $P_3(3.2) = 4 - 2(0.2) + 5(0.04) - \frac{1}{2}(0.008) = 4 - 0.4 + 0.2 - 0.004 = 3.796$. Lagrange with $n = 3$, $M = 6$:

$$|f(3.2) - P_3(3.2)| \le \frac{6}{4!}(0.2)^4 = \frac{6(0.0016)}{24} = 0.0004.$$

Hence $3.7956 \le f(3.2) \le 3.7964$.

Bài tập luyện tập

Let f be the function defined by the power series $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n\,3^n}$ for all x for which the series converges.
(a) Find the interval of convergence. Show the work that leads to your answer.
(b) Write the first three nonzero terms and the general term of the series for $f'(x)$. Find a closed-form expression for $f'(x)$ and the interval on which the series for f' converges.
(c) Use part (b) and the value $f(2)$ to find a closed-form expression for $f(x)$.
(d) Use the first two nonzero terms of the series to approximate $f(1)$. Show that this approximation differs from $f(1)$ by less than $\frac{1}{80}$.

Lời giải chi tiết

(a) $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{n}{n+1} \cdot \dfrac{|x-2|}{3} \to \dfrac{|x-2|}{3} < 1 \iff -1 < x < 5$. At $x = 5$: $\sum \frac{3^n}{n3^n} = \sum \frac{1}{n}$, the harmonic series, diverges. At $x = -1$: $\sum \frac{(-3)^n}{n3^n} = \sum \frac{(-1)^n}{n}$, converges by the Alternating Series Test. Interval: $[-1, 5)$.
(b) Differentiate term by term:

$$f'(x) = \sum_{n=1}^{\infty} \frac{(x-2)^{n-1}}{3^n} = \frac{1}{3} + \frac{x-2}{9} + \frac{(x-2)^2}{27} + \cdots + \frac{(x-2)^{n-1}}{3^n} + \cdots.$$

This is geometric with first term $\frac{1}{3}$ and ratio $\frac{x-2}{3}$: $f'(x) = \dfrac{1/3}{1 - \frac{x-2}{3}} = \dfrac{1}{5-x}$, converging for $|x-2| < 3$. At $x = -1$ the terms are $\frac{(-1)^{n-1}}{3}$, which do not approach 0, so the series for f' converges on $(-1, 5)$ — the endpoint $x = -1$ is lost.

(c) $f(2)=0$ (every term vanishes at the center). Then $f(x)=f(2)+\displaystyle\int_2^x \frac{dt}{5-t}=\left[-\ln(5-t)\right]_2^x=\ln 3-\ln(5-x)=\ln\frac{3}{5-x}$.

(d) $f(1)=\sum\frac{(-1)^n}{n3^n}=-\frac{1}{3}+\frac{1}{18}-\frac{1}{81}+\cdots$. First two terms: $-\frac{1}{3}+\frac{1}{18}=-\frac{5}{18}\approx -0.2778$. The series alternates and $\frac{1}{n3^n}$ decreases to 0, so the error is less than the first omitted term, $\frac{1}{3\cdot 27}=\frac{1}{81}<\frac{1}{80}$. (Check: $f(1)=\ln\frac{3}{4}\approx -0.2877$.)

Bài tập luyện tập

Let $f(x)=\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n^2}$.

(a) Find the interval of convergence of the series for f.

(b) Find the interval of convergence of the series for f'. Justify your treatment of each endpoint.

(c) Use the first two terms of the series for $f\left(-\frac{1}{2}\right)$ to approximate $f\left(-\frac{1}{2}\right)$, and find an upper bound for the error.

(d) Show that $f'(x)=-\dfrac{\ln(1-x)}{x}$ for $0<|x|<1$.

Lời giải chi tiết

(a) Ratio: $\left(\frac{n}{n+1}\right)^2|x|\to|x|$, so $R=1$. At $x=1$: $\sum\frac{1}{n^2}$, a convergent p-series. At $x=-1$: $\sum\frac{(-1)^n}{n^2}$ converges absolutely. Interval: $[-1,1]$.

(b) $f'(x)=\sum_{n=1}^{\infty}\frac{x^{n-1}}{n}$, radius 1. At $x=1$: $\sum\frac{1}{n}$ diverges. At $x=-1$: $\sum\frac{(-1)^{n-1}}{n}$ converges by the Alternating Series Test. Interval: $[-1,1)$.

(c) $f\left(-\frac{1}{2}\right)=-\frac{1}{2}+\frac{1}{4\cdot 4}-\frac{1}{8\cdot 9}+\cdots$. Two terms: $-\frac{1}{2}+\frac{1}{16}=-0.4375$. The terms $\frac{(1/2)^n}{n^2}$ decrease to 0 and alternate, so the error is less than $\frac{1}{72}\approx 0.0139$.

(d) For $|x|<1$, $\sum_{n=1}^{\infty}\frac{x^n}{n}=-\ln(1-x)$ (replace x by $-x$ in the series for $\ln(1+x)$ and multiply by -1). So $x\,f'(x)=\sum_{n=1}^{\infty}\frac{x^n}{n}=-\ln(1-x)$, and dividing by $x\neq 0$ gives $f'(x)=-\dfrac{\ln(1-x)}{x}$.

Bài tập luyện tập

The function f is defined by $f(x)=\dfrac{1-\cos x}{x^2}$ for $x\neq 0$ and $f(0)=\frac{1}{2}$.

(a) Write the first three nonzero terms and the general term of the Maclaurin series for f.

(b) Use the series to find $f'(0)$ and $f''(0)$. Does f have a relative maximum, a relative minimum, or neither at $x=0$? Justify.

(c) Use the first two nonzero terms of the series to approximate $\displaystyle\int_0^1 f(x)\,dx$. Show that the approximation differs from the exact value by less than 0.0003.

(d) Is the approximation in part (c) an overestimate or an underestimate? Explain.

Lời giải chi tiết

(a) $1-\cos x=\dfrac{x^2}{2!}-\dfrac{x^4}{4!}+\dfrac{x^6}{6!}-\cdots$. Dividing by x^2:

$$f(x)=\frac{1}{2}-\frac{x^2}{24}+\frac{x^4}{720}-\cdots+\frac{(-1)^n x^{2n}}{(2n+2)!}+\cdots.$$

(The constant term $\frac{1}{2}$ agrees with $f(0)$, so the series represents f at 0 as well.)

(b) There is no x term, so $f'(0)=0$. The coefficient of x^2 is $-\frac{1}{24}$, so $f''(0)=2!\cdot\left(-\frac{1}{24}\right)=-\frac{1}{12}<0$. By the Second Derivative Test, f has a **relative maximum** at $x=0$.

(c) $\displaystyle\int_0^1 f(x)\,dx=\left[\frac{x}{2}-\frac{x^3}{72}+\frac{x^5}{3600}-\cdots\right]_0^1=\frac{1}{2}-\frac{1}{72}+\frac{1}{3600}-\cdots$. Two terms: $\frac{1}{2}-\frac{1}{72}=\frac{35}{72}\approx 0.4861$. The series alternates with decreasing terms, so the error is less than $\frac{1}{3600}\approx 0.000278<0.0003$.

(d) The first omitted term, $+\frac{1}{3600}$, is positive, so the partial sum is less than the exact value: an **underestimate**.

Bài tập luyện tập

(a) Write the first four nonzero terms and the general term of the Maclaurin series for $g(x)=xe^{-x}$.

(b) Use the series in part (a) to express $\displaystyle\int_0^1 xe^{-x}\,dx$ as an infinite series. Write the first four terms and the general term.

(c) Evaluate $\displaystyle\int_0^1 xe^{-x}\,dx$ exactly using integration by parts, and hence find the exact sum of $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,(n+2)}$.

(d) What is the fewest number of terms of the series in part (b) needed to approximate the integral with error less than 0.01? Justify your answer.

Lời giải chi tiết

(a) $e^{-x}=\sum\frac{(-1)^n x^n}{n!}$, so

$$xe^{-x}=x-x^2+\frac{x^3}{2!}-\frac{x^4}{3!}+\cdots+\frac{(-1)^n x^{n+1}}{n!}+\cdots.$$

(b) $\displaystyle\int_0^1 xe^{-x}\,dx=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^1 x^{n+1}\,dx=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,(n+2)}=\frac{1}{2}-\frac{1}{3}+\frac{1}{8}-\frac{1}{30}+\cdots.$

(c) By parts with $u=x$, $dv=e^{-x}dx$: $\displaystyle\int_0^1 xe^{-x}\,dx=\left[-xe^{-x}\right]_0^1+\int_0^1 e^{-x}\,dx=-e^{-1}+(1-e^{-1})=1-\frac{2}{e}$. Therefore $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+2)}=1-\frac{2}{e}\approx 0.2642$.

(d) The terms $\frac{1}{n!(n+2)}$ are $\frac{1}{2},\frac{1}{3},\frac{1}{8},\frac{1}{30},\frac{1}{144},\ldots$, decreasing to 0, and the series alternates. Using the first k terms, the error is less than the $(k+1)$st term. The first term smaller than 0.01 is $\frac{1}{144}\approx 0.0069$ (the fifth), while $\frac{1}{30}\approx 0.033>0.01$. So **four terms** suffice: $\frac{1}{2}-\frac{1}{3}+\frac{1}{8}-\frac{1}{30}\approx 0.2583$, with error $<\frac{1}{144}<0.01$. Three terms are not guaranteed by the bound (their bound is $\frac{1}{30}$), and in fact $\left|\frac{1}{2}-\frac{1}{3}+\frac{1}{8}-\left(1-\frac{2}{e}\right)\right|\approx 0.027>0.01$.

Bài tập luyện tập

Let $h(x) = \arctan(2x)$.
(a) Write the first four nonzero terms and the general term of the Maclaurin series for h.
(b) Find the interval of convergence of this series. Justify.
(c) Use the first two nonzero terms to approximate $h(0.25) = \arctan(0.5)$, and find an upper bound for the error.
(d) Find $h^{(7)}(0)$.

Lời giải chi tiết

(a) Replace x by $2x$ in $\arctan x = \sum \frac{(-1)^n x^{2n+1}}{2n+1}$:

$$\arctan(2x) = 2x - \frac{8x^3}{3} + \frac{32x^5}{5} - \frac{128x^7}{7} + \cdots + \frac{(-1)^n 2^{2n+1} x^{2n+1}}{2n+1} + \cdots.$$

(b) The arctan series converges for $-1 \le u \le 1$, so we need $-1 \le 2x \le 1$: interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$. (Directly: at $x = \pm\frac{1}{2}$ the series is $\pm\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots\right)$, which converges by the Alternating Series Test; the ratio test gives $4x^2 < 1$ inside.)
(c) $h(0.25) \approx 2(0.25) - \frac{8}{3}(0.25)^3 = 0.5 - \frac{0.125}{3} \approx 0.45833$. At $x = 0.25$ the terms are $\frac{(0.5)^{2n+1}}{2n+1}$, which alternate and decrease to 0; the error is less than the next term, $\frac{32(0.25)^5}{5} = \frac{(0.5)^5}{5} = 0.00625$. (True value $\arctan 0.5 \approx 0.46365$.)
(d) $c_7 = -\frac{128}{7}$, so $h^{(7)}(0) = 7! \cdot \left(-\frac{128}{7}\right) = -128 \cdot 6! = -92160$.

Bài tập luyện tập

The Taylor series for a function f about $x = 1$ is given by

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, 2^n\, (x-1)^n}{n}$$

and converges to $f(x)$ on its interval of convergence.
(a) Find the interval of convergence.
(b) Find $f'''(1)$.
(c) The function g satisfies $g(1) = 3$ and $g'(x) = f(x)$. Write the first three nonzero terms of the Taylor series for g about $x = 1$.
(d) Identify f as a familiar function and evaluate $f(1.25)$ exactly.

Lời giải chi tiết

(a) $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n}{n+1} \cdot 2|x-1| \to 2|x-1| < 1 \iff \frac{1}{2} < x < \frac{3}{2}$. At $x = \frac{3}{2}$: $\sum \frac{(-1)^{n+1}}{n}$ converges (Alternating Series Test). At $x = \frac{1}{2}$: $\sum \frac{(-1)^{n+1}(-1)^n}{n} = -\sum \frac{1}{n}$ diverges. Interval: $\left(\frac{1}{2}, \frac{3}{2}\right]$.
(b) $c_3 = \frac{(-1)^4\, 2^3}{3} = \frac{8}{3}$, so $f'''(1) = 3! \cdot \frac{8}{3} = 16$.
(c) $f(x) = 2(x-1) - 2(x-1)^2 + \frac{8}{3}(x-1)^3 - \cdots$. Integrating from 1 and adding $g(1) = 3$:

$$g(x) = 3 + (x-1)^2 - \frac{2}{3}(x-1)^3 + \cdots.$$

(d) With $u = 2(x-1)$ the series is $\sum \frac{(-1)^{n+1}u^n}{n} = \ln(1+u)$, so $f(x) = \ln\left(1+2(x-1)\right) = \ln(2x-1)$. Then $f(1.25) = \ln 1.5 = \ln \frac{3}{2}$ ($x = 1.25$ is inside the interval).

Ôn tập nhanh — Unit 10:

- **Chọn phép thử hội tụ:** luôn thử *số hạng thứ* n trước (nếu $\lim a_n \neq 0$ thì phân kỳ ngay); chuỗi hình học hội tụ khi $|r| < 1$, tổng $\frac{a}{1-r}$ với a là *số hạng đầu tiên*; chuỗi p hội tụ khi $p > 1$; có giai thừa hoặc mũ n thì dùng phép thử tỉ số; phân thức đa thức thì so sánh giới hạn với chuỗi p.
- **Hội tụ tuyệt đối / có điều kiện:** xét $\sum |a_n|$ trước; nếu $\sum |a_n|$ phân kỳ nhưng chuỗi đan dấu thoả điều kiện (giảm dần về 0) thì hội tụ có điều kiện. Sai số chuỗi đan dấu nhỏ hơn số hạng đầu tiên bị bỏ, cùng dấu với số hạng đó.
- **Đa thức Taylor:** hệ số của $(x-a)^k$ là $\frac{f^{(k)}(a)}{k!}$; ngược lại $f^{(k)}(a) = k!\,c_k$. Viết theo lũy thừa của $(x-a)$, không phải của x.
- **Chặn sai số Lagrange:** $|f(x) - P_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$, M là chặn trên của $|f^{(n+1)}|$ trên cả đoạn giữa a và x. Từ đó suy ra khoảng chứa $f(x)$.
- **Khoảng hội tụ:** phép thử tỉ số cho bán kính, sau đó *bắt buộc* thay từng đầu mút vào chuỗi gốc. Đạo hàm/tích phân từng hạng giữ nguyên bán kính nhưng đầu mút có thể thay đổi.
- **Chuỗi phải thuộc:** e^x, $\sin x$, $\cos x$ (mọi x); $\frac{1}{1-x}$ trên $(-1,1)$; $\ln(1+x)$ trên $(-1,1]$; $\arctan x$ trên $[-1,1]$. Lấy chuỗi mới bằng thế, nhân x^k, đạo hàm, tích phân — không cần tính đạo hàm cấp cao.
- **Ứng dụng chuỗi:** tích phân không sơ cấp (tích phân từng hạng + chặn sai số đan dấu), tổng chuỗi số (nhận ra chuỗi quen thuộc, kiểm tra x trong khoảng hội tụ và chỉ số bắt đầu), giới hạn dạng $\frac{0}{0}$.
- **Trình bày FRQ:** nêu tên phép thử đã dùng, kiểm tra điều kiện của phép thử, ghi rõ giá trị M hoặc số hạng bị bỏ khi chặn sai số; đáp số máy tính làm tròn 3 chữ số thập phân.

AP Calculus BC Practice Exam

Mục tiêu Unit: Làm đề này như một buổi thi thật: chuẩn bị đồng hồ, giấy nháp và máy tính đồ thị; làm **lần lượt bốn phần, đúng thời gian ghi ở đầu mỗi phần**, không xem lời giải và chỉ dùng máy tính ở Part B của Section I và Part A của Section II. Làm xong, đối chiếu Section I với **bảng đáp án**, đọc lời giải của *mọi* câu sai và cả những câu đúng nhờ đoán; chấm Section II theo **thang điểm 9 điểm/câu** ở cuối chương — chỉ cho điểm khi bài làm có đúng những gì thang điểm yêu cầu (thiết lập tích phân, lý do, đơn vị). Cuối cùng quy đổi điểm thô sang thang 1–5 để biết mình đang ở đâu và ôn lại các Unit còn yếu.

11.1 Exam Overview and Directions

The AP Calculus BC Exam has two sections, each worth half of the exam score. Section I is multiple choice and Section II is free response; each section is split into a part that is taken *without* a calculator and a part that *requires* a graphing calculator. This practice exam follows that structure exactly.

Section	Part	Questions	Time	Calculator	Share of score
I: Multiple choice	Part A	30 (Q1–30)	60 min	Not permitted	33.3%
	Part B	15 (Q31–45)	45 min	Graphing calculator required	16.7%
II: Free response	Part A	2 (FRQ 1–2)	30 min	Graphing calculator required	16.7%
	Part B	4 (FRQ 3–6)	60 min	Not permitted	33.3%

Total testing time: 3 hours 15 minutes. Each multiple-choice question has four answer choices, (A)–(D), and there is no penalty for a wrong answer.

Calculator capabilities you may need. On the calculator parts you are expected to (1) graph a function in an arbitrary window, (2) find the zeros of a function (solve equations numerically), (3) compute the derivative of a function numerically at a point, and (4) compute a definite integral numerically. In free response, write the *setup* (the integral, the derivative, the equation) in standard mathematical notation, not calculator syntax, and then give the value.

Answer conventions used throughout this exam. Unless a question says otherwise: the domain of a function f is the set of all real numbers x for which $f(x)$ is a real number; $\sin^{-1} x$ and $\arcsin x$ both denote the inverse sine; and on calculator questions a decimal answer must be correct to **three places after the decimal point** (rounded or truncated). If the exact value of a calculator answer is not one of the choices, choose the choice that best approximates it.

VN GIẢI THÍCH TIẾNG VIỆT

Cách làm đề giống thi thật. Bấm giờ riêng cho từng phần và dừng bút khi hết giờ — kể cả khi còn câu chưa làm (đánh dấu lại để luyện sau, nhưng *không* tính điểm). Trắc nghiệm không trừ điểm câu sai, nên trước khi hết giờ hãy tô đáp án cho mọi câu. Ở Section II, giám khảo chấm **quá trình**: một con số "trơ trọi" không kèm tích phân hay lập luận thường được 0 điểm, còn câu hỏi có chữ *justify* bắt buộc phải nêu lý do bằng lời (ví dụ "g' đổi dấu từ dương sang âm tại $x = 5$"). Khi Section II Part A (có máy tính) kết thúc, bạn chuyển sang Part B và cất máy tính;

theo quy định hiện hành, trong thời gian Part B bạn vẫn được quay lại viết tiếp Part A nhưng không được dùng máy tính.

11.2 Section I, Part A: Multiple Choice (No Calculator)

Section I, Part A **30 questions** **Time: 60 minutes** **NO CALCULATOR**
Directions: Solve each problem. After examining the form of the choices, decide which is the best of the choices given. Do not spend too much time on any one problem. A calculator may **not** be used on this part of the exam.

1. $\lim\limits_{x\to 0} \dfrac{x\sin x}{1-\cos(2x)}$ is

(A) $\dfrac{1}{2}$
(B) 1
(C) 2
(D) nonexistent

2. Let f be the function defined by $f(x) = \begin{cases} kx+3, & x<2 \\ x^2-k, & x\geq 2. \end{cases}$ For what value of k is f continuous at $x=2$?

(A) $-\dfrac{1}{3}$
(B) $\dfrac{1}{4}$
(C) $\dfrac{1}{3}$
(D) 1

3. If $f(x) = x^2 \ln x$, then $f'(e) =$

(A) 2
(B) $2e$
(C) e^2
(D) $3e$

4. $\lim\limits_{h\to 0} \dfrac{\tan(\frac{\pi}{4}+h)-1}{h}$ is

(A) 0
(B) 1
(C) 2
(D) nonexistent

5. The table below gives values of the differentiable functions f and g and their derivatives at selected values of x. If $h(x) = f(g(x))$, what is $h'(1)$?

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	3	-2	2	4
2	1	5	1	-3

(A) -15
(B) -8
(C) 12
(D) 20

6. What is the slope of the line tangent to the curve $x^2+xy+y^3=3$ at the point $(1,1)$?

(A) -1
(B) $-\dfrac{3}{4}$
(C) $-\dfrac{1}{2}$
(D) $\dfrac{3}{4}$

7. Let $f(x) = x^3 + 2x + 1$, and let g be the inverse function of f. What is the value of $g'(4)$?

(A) $\frac{1}{5}$
(B) 1
(C) 5
(D) 50

8. A particle moves along the x-axis so that its position at time $t \geq 0$ is $x(t) = t^3 - 6t^2 + 9t$. For which values of t is the speed of the particle increasing?

(A) $1 < t < 2$ and $t > 3$
(B) $0 < t < 1$ and $t > 3$
(C) $t > 2$ only
(D) $1 < t < 3$ only

9. $\lim_{x \to 0} \frac{e^{2x} - 1 - 2x}{x^2}$ is

(A) 0
(B) $\frac{1}{2}$
(C) 1
(D) 2

10. Let $f(x) = x^3 - 3x$. What value of c satisfies the conclusion of the Mean Value Theorem for f on the closed interval $[0, 2]$?

(A) 1
(B) $\frac{2}{\sqrt{3}}$
(C) $\sqrt{3}$
(D) 2

11. The derivative of a function f is given by $f'(x) = x(x-2)^2(x+3)$. At which value(s) of x does f have a relative minimum?

(A) $x = -3$ only
(B) $x = 0$ only
(C) $x = 0$ and $x = 2$
(D) $x = -3$ and $x = 2$

12. What is the absolute maximum value of $f(x) = xe^{-x}$ on the closed interval $[0, 3]$?

(A) 0
(B) $\frac{3}{e^3}$
(C) $\frac{1}{e}$
(D) 1

13. $\frac{d}{dx} \int_1^{x^2} \cos(t^3)\, dt =$

(A) $2x \cos(x^6)$
(B) $\cos(x^6)$
(C) $2x \cos(x^3)$
(D) $\cos(x^6) - \cos 1$

14. $\int_0^{\pi/2} \sin x \cos^2 x \, dx =$

(A) $-\frac{1}{3}$
(B) 0
(C) $\frac{1}{6}$
(D) $\frac{1}{3}$

15. $\int_0^1 xe^x\,dx =$

(A) 1
(B) $e-1$
(C) e
(D) $2e-1$

16. $\int_3^5 \frac{1}{x^2-x-2}\,dx =$

(A) $\frac{1}{3}\ln 2$
(B) $\frac{1}{3}\ln 3$
(C) $\ln 2$
(D) $3\ln 2$

17. $\int_2^\infty \frac{1}{x(\ln x)^2}\,dx$ is

(A) $\ln 2$
(B) $\frac{1}{\ln 2}$
(C) $\frac{1}{(\ln 2)^2}$
(D) divergent

18. If $\int_1^4 f(x)\,dx = 7$ and $\int_1^2 f(x)\,dx = 3$, then $\int_2^4 (2f(x)+1)\,dx =$

(A) 8
(B) 9
(C) 10
(D) 14

19. Let $y = f(x)$ be the solution to the differential equation $\frac{dy}{dx} = xy^2$ with initial condition $f(0) = 1$. What is $f(1)$?

(A) $\frac{1}{2}$
(B) $\frac{2}{3}$
(C) $e^{1/2}$
(D) 2

20. Let $y = f(x)$ be the solution to $\frac{dy}{dx} = x + y$ with $f(0) = 1$. What is the approximation for $f(1)$ obtained by using Euler's method with two steps of equal size, starting at $x = 0$?

(A) 1.25
(B) 2
(C) 2.5
(D) $2e-2$

21. What is the area of the region bounded by the graphs of $y = x^2$ and $y = 2x$?

(A) $\frac{4}{3}$
(B) $\frac{8}{3}$
(C) 4
(D) $\frac{20}{3}$

22. The region bounded by the graph of $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the x-axis. What is the volume of the solid generated?

(A) 2π
(B) 4π
(C) $\frac{16\pi}{3}$
(D) 8π

23. A curve is defined by the parametric equations $x = t^2 + 1$ and $y = t^3 - 3t$. What is the slope of the line tangent to the curve at the point where $t = 2$?

(A) $\frac{4}{9}$
(B) $\frac{3}{2}$
(C) $\frac{9}{4}$
(D) 9

24. What is the area enclosed by one petal of the polar curve $r = \sin(3\theta)$?

(A) $\frac{\pi}{12}$
(B) $\frac{\pi}{6}$
(C) $\frac{\pi}{4}$
(D) $\frac{\pi}{3}$

25. A particle moves in the xy-plane with position vector $\langle t^2,\ \sin(\pi t)\rangle$ at time t. What is the speed of the particle at time $t = 1$?

(A) 2
(B) π
(C) $\sqrt{4 + \pi^2}$
(D) $2 + \pi$

26. What is the sum of the series $\sum_{n=0}^{\infty} 3\left(-\frac{2}{5}\right)^n$?

(A) $-\frac{6}{7}$
(B) $\frac{15}{7}$
(C) 5
(D) The series diverges.

27. Which of the following series converges conditionally?

(A) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$
(B) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$
(C) $\sum_{n=1}^{\infty} \frac{(-1)^n\, n}{n+1}$
(D) $\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$

28. What is the coefficient of x^6 in the Maclaurin series for $x^2 e^{-x^2}$?

(A) $-\frac{1}{2}$
(B) $-\frac{1}{6}$
(C) $\frac{1}{24}$
(D) $\frac{1}{2}$

29. What is the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x+1)^n}{n^2 \cdot 4^n}$?

(A) $-5 \le x \le 3$
(B) $-5 < x < 3$
(C) $-5 \le x < 3$
(D) $-3 \le x \le 5$

30. A function f has derivatives of all orders, with $f(1) = 2$, $f'(1) = -3$, $f''(1) = 4$, and $f'''(1) = 12$. What is the approximation of $f(1.5)$ given by the third-degree Taylor polynomial for f about $x = 1$?

(A) 1.00
(B) 1.25
(C) 1.75
(D) 3.00

END OF PART A. If you finish before time is called, you may check your work on Part A only.

11.3 Section I, Part B: Multiple Choice (Calculator Required)

Section I, Part B **15 questions** **Time: 45 minutes** **GRAPHING CALCULATOR REQUIRED**
Directions: Solve each problem and choose the best answer. A graphing calculator is required for some questions on this part of the exam. If the exact numerical value of the correct answer does not appear among the choices, select the choice that best approximates the exact value.

31. The function f is continuous on the closed interval $[1, 5]$ and has the values given in the table. What is the least number of zeros f must have on $[1, 5]$?

x	1	2	3	4	5
$f(x)$	4	-1	2	-3	1

(A) 1
(B) 2
(C) 3
(D) 4

32. Let $f(x) = \ln(x+2)$ and $g(x) = x^2 - 1$. At what value of x do the graphs of f and g have parallel tangent lines?

(A) -2.225
(B) 0.225
(C) 0.414
(D) 1.501

33. A particle moves along the x-axis with velocity $v(t) = t\sin\left(\frac{t^2}{2}\right)$ for $t \geq 0$. What is the acceleration of the particle at time $t = 2$?

(A) -1.665
(B) -0.755
(C) 0.077
(D) 1.819

34. The function f is differentiable, and $f'(x) = e^{-x/3}\sin(x^2)$. On the interval $0 < x < 3.5$, at which values of x does f have a relative maximum?

(A) 1.772 and 3.070
(B) 2.507 only
(C) 1.772 only
(D) 1.772, 2.507, and 3.070

35. Water flows into a tank at the rate $R(t) = 20 + 10\sin\left(\frac{t}{3}\right)$ gallons per hour, where t is measured in hours. No water leaves the tank. If the tank holds 100 gallons at time $t = 0$, how many gallons does it hold at time $t = 6$?

(A) 129.093
(B) 162.484
(C) 220.000
(D) 262.484

36. The function f is continuous, and selected values are given in the table. What is the trapezoidal sum approximation of $\displaystyle\int_0^{10} f(t)\,dt$ using the four subintervals indicated by the table?

t	0	2	5	6	10
$f(t)$	3	7	4	8	2

(A) 42
(B) 52.5
(C) 63
(D) 105

37. A quantity y changes according to $\dfrac{dy}{dt} = ky$, where k is a constant. If $y(0) = 50$ and $y(3) = 80$, what is $y(10)$?

(A) 239.536
(B) 369.453
(C) 383.257
(D) 5497.558

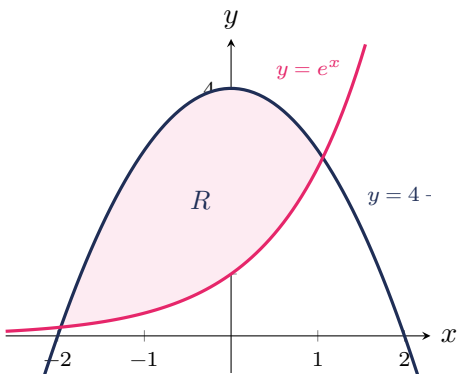
38. A population $P(t)$ satisfies the logistic differential equation $\dfrac{dP}{dt} = 0.3P\left(1 - \dfrac{P}{1200}\right)$ with $P(0) = 200$, so that $P(t) = \dfrac{1200}{1 + 5e^{-0.3t}}$. At what time t is the population growing fastest?

(A) 0.483
(B) 1.609
(C) 5.365
(D) 5.973

39. A particle moves along a line with velocity $v(t) = (t-1)e^{\sin t}$. What is the total distance traveled by the particle over the time interval $0 \le t \le 3$?

(A) 2.303
(B) 3.229
(C) 3.932
(D) 4.635

40. Let R be the region bounded by the graphs of $y = 4 - x^2$ and $y = e^x$, shown in the figure. What is the volume of the solid generated when R is revolved about the x-axis?



(A) 6.428
(B) 32.629
(C) 51.374
(D) 84.705

41. A particle moves in the xy-plane so that its velocity vector at time t is $\langle \cos(t^2),\ e^{t/2} \rangle$. What is the total distance traveled by the particle over the time interval $0 \le t \le 2$?

(A) 2.796
(B) 3.437
(C) 3.828
(D) 4.930

42. A polar curve is given by $r = 2 + \sin(3\theta)$. For the point on the curve where $\theta = 1$, what is the value of $\dfrac{dy}{d\theta}$?

(A) -3.406
(B) -2.970
(C) -1.342
(D) 0.394

43. The fourth-degree Taylor polynomial for a function f about $x = 0$ is used to approximate $f(0.5)$. If $\left|f^{(5)}(x)\right| \le 12$ for $0 \le x \le 0.5$, which of the following is the upper bound for the error of this approximation given by the Lagrange error bound?

(A) $\dfrac{1}{3840}$
(B) $\dfrac{1}{320}$
(C) $\dfrac{1}{32}$
(D) $\dfrac{3}{8}$

44. Let $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ for $-1 \le x \le 1$. What is the value of $f'(0.5)$?

(A) 0.582
(B) 0.693
(C) 1.386
(D) 2.000

45. The nth partial sum S_n of the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3}$ is used to approximate the sum of the series. According to the alternating series error bound, what is the least value of n for which the error is guaranteed to be less than 0.0005?

(A) 10
(B) 11
(C) 12
(D) 13

END OF SECTION I.

11.4 Section II, Part A: Free Response (Calculator Required)

Section II, Part A **2 questions** **Time: 30 minutes** **GRAPHING CALCULATOR REQUIRED**

Directions: Show all of your work. Clearly label any functions, graphs, tables, or other objects that you use. Your work will be scored on the correctness and completeness of your methods as well as your answers; answers without supporting work will usually not receive credit. Justifications require that you give mathematical reasons. Write integrals and derivatives in standard notation (for example $\int_1^5 x^2\,dx$), not calculator syntax. Unless otherwise specified, answers (numeric or algebraic) need not be simplified, and decimal answers must be correct to three places after the decimal point.

Question 1

t (hours)	0	2	5	7	10	12
$R(t)$ (gallons per hour)	130	150	170	165	150	120

Water is pumped into a storage tank at a rate modeled by the differentiable function R, where $R(t)$ is measured in gallons per hour and t is measured in hours for $0 \le t \le 12$. Selected values of $R(t)$ are given in the table above. During the same time interval, water is drained from the tank at the rate $D(t) = 80 + 120\sin\left(\frac{t}{4}\right)$ gallons per hour. At time $t = 0$ the tank contains 500 gallons of water.

(a) Use the data in the table to estimate $R'(6)$. Show the computations that lead to your answer, and indicate units of measure.

(b) Use a trapezoidal sum with the five subintervals indicated by the table to approximate $\int_0^{12} R(t)\,dt$. Using correct units, explain the meaning of $\int_0^{12} R(t)\,dt$ in the context of the problem.

(c) Must there be a value of c, for $0 < c < 12$, such that $R'(c) = 0$? Justify your answer.

(d) Using your approximation from part (b), find the amount of water in the tank at time $t = 12$.

(e) Is the amount of water in the tank increasing or decreasing at time $t = 7$? Give a reason for your answer.

Question 2

A particle moves in the xy-plane so that its velocity vector at time t, for $0 \le t \le 3$, is

$$\langle x'(t),\ y'(t)\rangle = \left\langle 1 + 2\sin(t^2),\ e^{t/2}\cos t \right\rangle.$$

At time $t = 0$ the particle is at the point $(2, -1)$.

(a) Find the speed of the particle at time $t = 1.5$, and find the acceleration vector of the particle at time $t = 1.5$.

(b) For $0 < t < 3$, find all intervals of time on which the particle is moving to the left. Give a reason for your answer.

(c) Find the position of the particle at time $t = 3$.

(d) Find the total distance traveled by the particle over the time interval $0 \le t \le 3$.

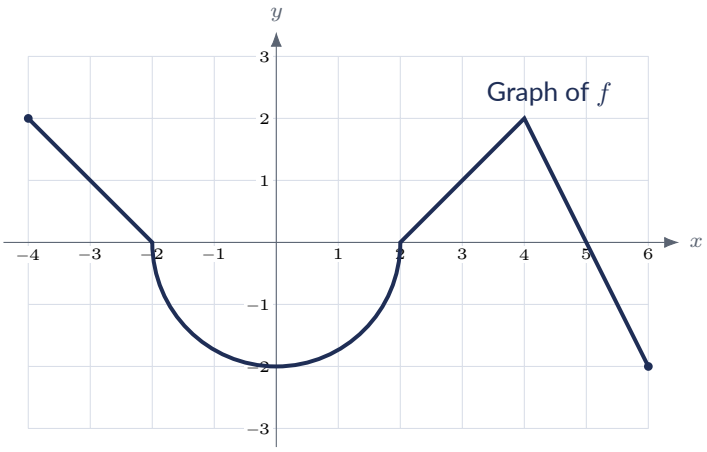
END OF PART A. Put your calculator away before beginning Part B.

11.5 Section II, Part B: Free Response (No Calculator)

Section II, Part B **4 questions** **Time: 60 minutes** **NO CALCULATOR**

Directions: A calculator is **not** permitted. Show all of your work; justifications require mathematical reasons. During this part you may continue to work on the questions in Part A, without the use of a calculator.

Question 3



The graph of f on $-4 \le x \le 6$ consists of three line segments and a semicircle of radius 2 centered at the origin.

The continuous function f is defined on the closed interval $-4 \le x \le 6$. The graph of f, shown above, consists of a line segment from $(-4, 2)$ to $(-2, 0)$, the lower half of the circle $x^2 + y^2 = 4$, a line segment from $(2, 0)$ to $(4, 2)$, and a line segment from $(4, 2)$ to $(6, -2)$. Let g be the function given by $g(x) = \displaystyle\int_0^x f(t)\,dt$.

(a) Find $g(-4)$ and $g(6)$.

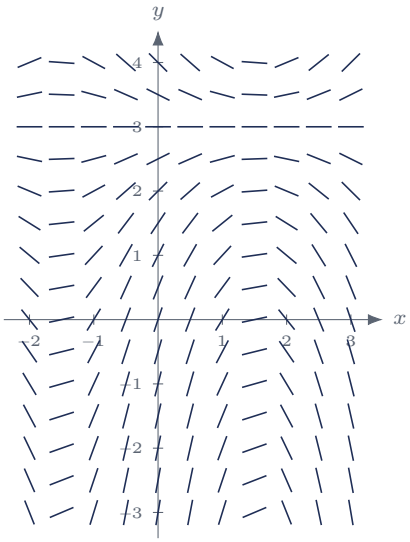
(b) On the open interval $-4 < x < 6$, find all values of x at which g has a relative maximum. Justify your answer.

(c) Find the absolute minimum value of g on the closed interval $-4 \le x \le 6$. Justify your answer.

(d) Find $\displaystyle\lim_{x \to 0} \frac{g(x)}{\sin x}$, or show that it does not exist.

Question 4

Consider the differential equation $\frac{dy}{dx} = (3-y)\cos x$. Let $y = f(x)$ be the particular solution to the differential equation with initial condition $f(0) = 1$.



Slope field for $\frac{dy}{dx} = (3-y)\cos x$ at the points $x = -2, -1.5, \ldots, 3$ and $y = -3, -2.5, \ldots, 4$.

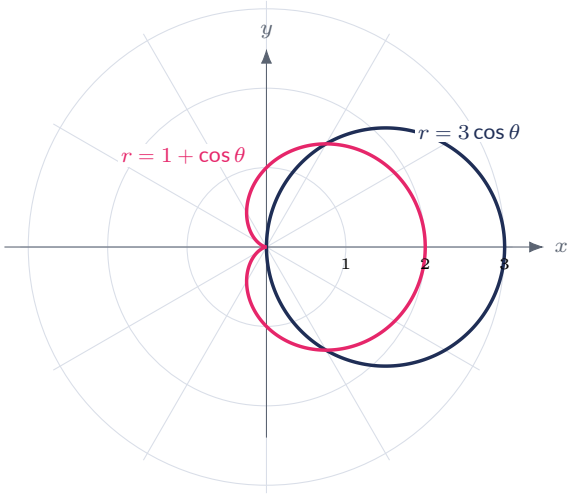
(a) On the slope field shown, sketch the solution curve that passes through the point $(0, 1)$.

(b) Write an equation for the line tangent to the graph of f at $x = 0$, and use it to approximate $f(0.2)$.

(c) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Is the approximation in part (b) an overestimate or an underestimate of $f(0.2)$? Give a reason for your answer.

(d) Use separation of variables to find $y = f(x)$, the particular solution with $f(0) = 1$.

Question 5



The circle $r = 3\cos\theta$ and the cardioid $r = 1 + \cos\theta$.

The figure shows the graphs of the polar curves $r = 3\cos\theta$ and $r = 1+\cos\theta$. The curves intersect at the pole and at two other points.

(a) Let R be the region that is inside the graph of $r = 3\cos\theta$ and outside the graph of $r = 1 + \cos\theta$. Find the area of R.

(b) Find the area of the region that is inside *both* graphs.

(c) Find the slope of the line tangent to the graph of $r = 1 + \cos\theta$ at the point where $\theta = \dfrac{\pi}{2}$.

(d) A particle moves along the cardioid $r = 1 + \cos\theta$ so that $\dfrac{d\theta}{dt} = 3$ for all times t. Find the value of $\dfrac{dr}{dt}$ at the instant when $\theta = \dfrac{\pi}{3}$, and interpret the meaning of this value in terms of the motion of the particle.

Question 6

The function f is defined by the power series

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n \cdot 3^n} = \frac{x-2}{3} - \frac{(x-2)^2}{2 \cdot 3^2} + \frac{(x-2)^3}{3 \cdot 3^3} - \cdots + \frac{(-1)^{n+1}(x-2)^n}{n \cdot 3^n} + \cdots$$

for all real numbers x for which the series converges.

(a) Find the interval of convergence of the power series for f. Justify your answer.

(b) Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 2$.

(c) Use the first two nonzero terms of the power series for f to approximate $f(3)$. Show that this approximation differs from $f(3)$ by less than $\frac{1}{50}$.

(d) Let $g(x) = \int_2^x f(t)\,dt$. Write the first two nonzero terms and the general term of the Taylor series for g about $x = 2$.

END OF EXAM.

11.6 Answer Key

Q	Ans	Q	Ans	Q	Ans	Q	Ans	Q	Ans
1	A	10	B	19	D	28	D	37	A
2	C	11	B	20	C	29	A	38	C
3	D	12	C	21	A	30	B	39	D
4	C	13	A	22	D	31	D	40	D
5	D	14	D	23	C	32	B	41	C
6	B	15	A	24	A	33	B	42	C
7	A	16	A	25	C	34	A	43	B
8	A	17	B	26	B	35	D	44	C
9	D	18	C	27	B	36	B	45	C

Questions 1–30: Part A (no calculator). Questions 31–45: Part B (calculator required).

Unit (Course and Exam Description)	Multiple-choice questions	Free response
1 Limits and Continuity	1, 2, 31	—
2 Differentiation: Definition and Fundamental Properties	3, 4, 32	1(a)
3 Differentiation: Composite, Implicit, and Inverse Functions	5, 6, 7	—
4 Contextual Applications of Differentiation	8, 9, 33	3(d), 4(b), 5(d)
5 Analytical Applications of Differentiation	10, 11, 12, 34	1(c), 3(b)(c)
6 Integration and Accumulation of Change	13–18, 35, 36	1(b)(d), 3(a)
7 Differential Equations	19, 20, 37, 38	4
8 Applications of Integration	21, 22, 39, 40	1(d)
9 Parametric Equations, Polar Coordinates, Vector-Valued Functions	23, 24, 25, 41, 42	2, 5
10 Infinite Sequences and Series	26–30, 43, 44, 45	6

Use this table to find the units behind your mistakes. The multiple-choice distribution follows the BC weightings: Units 6 and 10 carry the most questions, and Unit 9 about 11%.

11.7 Multiple-Choice Solutions

Part A (no calculator)

Question 1 Answer (A)

Use $1-\cos(2x)=2\sin^2 x$:

$$\lim_{x\to 0}\frac{x\sin x}{2\sin^2 x}=\lim_{x\to 0}\frac{1}{2}\cdot\frac{x}{\sin x}=\frac{1}{2}\cdot 1=\frac{1}{2}.$$

Choice (C) is the reciprocal, from inverting the fraction; (B) comes from approximating $1-\cos 2x$ by x^2, i.e. losing the factor 2 in $2\sin^2 x$.

Question 2 Answer (C)

Continuity at $x=2$ requires the left-hand limit to equal $f(2)$: $2k+3=2^2-k \Rightarrow 3k=1 \Rightarrow k=\frac{1}{3}$. Choice (B) results from the slip $2k+3=4-2k$, and (D) from the sign slip $2k+3=4+k$.

Question 3 Answer (D)

Product rule: $f'(x) = 2x\ln x + x^2 \cdot \dfrac{1}{x} = 2x\ln x + x$. Then $f'(e) = 2e(1) + e = 3e$. Choice (B) drops the second term of the product rule; (A) multiplies the derivatives, $(2x)(\frac{1}{x}) = 2$; (C) is $f(e)$.

Question 4 Answer (C)

Since $\tan\frac{\pi}{4} = 1$, the expression is the difference quotient $\dfrac{\tan(\frac{\pi}{4}+h) - \tan\frac{\pi}{4}}{h}$, so the limit is the derivative of $\tan x$ at $x = \frac{\pi}{4}$:

$$\sec^2\left(\frac{\pi}{4}\right) = \left(\sqrt{2}\right)^2 = 2.$$

Choice (B) is the value of $\tan\frac{\pi}{4}$, not its derivative.

Question 5 Answer (D)

Chain rule: $h'(1) = f'(g(1)) \cdot g'(1) = f'(2) \cdot g'(1) = 5 \cdot 4 = 20$. Choice (B) uses $f'(1)g'(1)$ (evaluating f' at the wrong input); (A) uses $f'(2)g'(2)$; (C) uses $f(1)g'(1)$.

Question 6 Answer (B)

Differentiate implicitly, using the product rule on xy:

$$2x + \left(y + x\frac{dy}{dx}\right) + 3y^2\frac{dy}{dx} = 0.$$

At $(1,1)$: $2 + 1 + \frac{dy}{dx} + 3\frac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -\dfrac{3}{4}$. Choice (A) forgets the $x\frac{dy}{dx}$ term; (C) treats the derivative of xy as simply $\frac{dy}{dx}$.

Question 7 Answer (A)

First find the input: $f(1) = 1 + 2 + 1 = 4$, so $g(4) = 1$. Since $f'(x) = 3x^2 + 2$, $f'(1) = 5$, and

$$g'(4) = \frac{1}{f'(g(4))} = \frac{1}{f'(1)} = \frac{1}{5}.$$

Choice (B) is $g(4)$; (C) is $f'(1)$ without the reciprocal; (D) is $f'(4)$, which evaluates f' at the wrong point.

Question 8 Answer (A)

$v(t) = 3t^2 - 12t + 9 = 3(t-1)(t-3)$ and $a(t) = 6t - 12 = 6(t-2)$. Speed increases when v and a have the same sign.

interval	$0 < t < 1$	$1 < t < 2$	$2 < t < 3$	$t > 3$
sign of v	$+$	$-$	$-$	$+$
sign of a	$-$	$-$	$+$	$+$
speed	decreasing	**increasing**	decreasing	**increasing**

Choice (B) lists where $v > 0$ (moving right), and (C) where $a > 0$; neither alone decides whether

speed increases.

Question 9 Answer (D)

The form is $\frac{0}{0}$. L'Hospital's Rule twice:

$$\lim_{x\to 0}\frac{2e^{2x}-2}{2x}\left(\tfrac{0}{0}\right)=\lim_{x\to 0}\frac{4e^{2x}}{2}=2.$$

(Check with series: $e^{2x}-1-2x=\frac{(2x)^2}{2!}+\cdots=2x^2+\cdots$.) Stopping after one application and substituting gives the indeterminate $\frac{0}{0}$, not 0, so (A) is wrong.

Question 10 Answer (B)

f is a polynomial, so the MVT applies. The average rate of change is $\dfrac{f(2)-f(0)}{2-0}=\dfrac{2-0}{2}=1$. Solve $f'(c)=3c^2-3=1$: $c^2=\frac{4}{3}$, and $c=\dfrac{2}{\sqrt{3}}\approx 1.155$ lies in $(0,2)$. Choice (A) is where $f'(c)=0$ (Rolle's conclusion, which does not apply because $f(0)\neq f(2)$); (C) is a zero of f.

Question 11 Answer (B)

The critical points are $x=-3,0,2$. Because $(x-2)^2\geq 0$, the sign of f' is the sign of $x(x+3)$, which is $+$ on $x<-3$, $-$ on $-3<x<0$, and $+$ for $x>0$ (except $f'(2)=0$).



f' changes from negative to positive only at $x=0$, so the only relative minimum is at $x=0$. At $x=-3$, f' changes from positive to negative (relative maximum), and at $x=2$, f' does not change sign (no extremum) — the double root is the trap in (C) and (D).

Question 12 Answer (C)

$f'(x)=e^{-x}-xe^{-x}=(1-x)e^{-x}$, which is 0 only at $x=1$. Candidates test: $f(0)=0$, $f(1)=\frac{1}{e}\approx 0.368$, $f(3)=\frac{3}{e^3}\approx 0.149$. The absolute maximum value is $\frac{1}{e}$. Choice (D) is the x-coordinate where the maximum occurs, not the maximum value.

Question 13 Answer (A)

By the Fundamental Theorem of Calculus with the chain rule, $\dfrac{d}{dx}\displaystyle\int_1^{x^2}\cos(t^3)\,dt=\cos((x^2)^3)\cdot 2x=2x\cos(x^6)$. Choice (B) forgets the factor $\frac{d}{dx}(x^2)=2x$; (D) wrongly includes the constant lower limit, whose contribution is 0.

Question 14 Answer (D)

Let $u=\cos x$, $du=-\sin x\,dx$; $x=0\mapsto u=1$, $x=\frac{\pi}{2}\mapsto u=0$:

$$\int_0^{\pi/2}\sin x\cos^2 x\,dx=\int_1^0 -u^2\,du=\int_0^1 u^2\,du=\frac{1}{3}.$$

Choice (A) comes from losing the minus sign of du (or not reversing the limits).

Question 15 Answer (A)

Integration by parts with $u = x$, $dv = e^x\,dx$:

$$\int_0^1 xe^x\,dx = \Big[xe^x\Big]_0^1 - \int_0^1 e^x\,dx = e - (e-1) = 1.$$

Choice (D) results from *adding* $\int v\,du$ instead of subtracting: $[xe^x + e^x]_0^1 = 2e - 1$.

Question 16 Answer (A)

$x^2 - x - 2 = (x-2)(x+1)$, and $\dfrac{1}{(x-2)(x+1)} = \dfrac{1/3}{x-2} - \dfrac{1/3}{x+1}$. Then

$$\int_3^5 \frac{dx}{x^2-x-2} = \frac{1}{3}\Big[\ln|x-2| - \ln|x+1|\Big]_3^5 = \frac{1}{3}\Big[(\ln 3 - \ln 6) - (\ln 1 - \ln 4)\Big] = \frac{1}{3}\ln\frac{3\cdot 4}{6} = \frac{1}{3}\ln 2.$$

Choice (B) keeps only the $\ln|x-2|$ term; (C) drops the partial-fraction coefficient $\frac{1}{3}$.

Question 17 Answer (B)

Let $u = \ln x$, $du = \frac{dx}{x}$:

$$\int_2^\infty \frac{dx}{x(\ln x)^2} = \lim_{b\to\infty}\Big[-\frac{1}{\ln x}\Big]_2^b = \lim_{b\to\infty}\Big(-\frac{1}{\ln b} + \frac{1}{\ln 2}\Big) = \frac{1}{\ln 2}.$$

The integral converges (unlike $\int_2^\infty \frac{dx}{x\ln x}$, which diverges), so (D) is wrong.

Question 18 Answer (C)

$\int_2^4 f(x)\,dx = 7 - 3 = 4$, so $\int_2^4 (2f(x)+1)\,dx = 2(4) + \int_2^4 1\,dx = 8 + 2 = 10$. Choice (A) forgets the integral of the constant 1; (B) adds 1 instead of $1\cdot(4-2)$.

Question 19 Answer (D)

Separate: $y^{-2}\,dy = x\,dx \;\Rightarrow\; -\dfrac{1}{y} = \dfrac{x^2}{2} + C$. With $y(0) = 1$: $C = -1$. So $\dfrac{1}{y} = 1 - \dfrac{x^2}{2}$ and $y = \dfrac{2}{2-x^2}$ (valid for $|x| < \sqrt{2}$). Then $f(1) = \dfrac{2}{1} = 2$. Choice (B) comes from dropping the minus sign in $\int y^{-2}dy = -y^{-1}$; (C) solves $\frac{dy}{dx} = xy$ instead.

Question 20 Answer (C)

Step size $\Delta x = 0.5$.

x	y	slope $x+y$	$\Delta y = 0.5\cdot$slope
0	1	1	0.5
0.5	1.5	2	1
1	2.5		

Choice (B) uses one step of size 1; (D) is the exact value $2e-2$ of the solution $y = 2e^x - x - 1$, not Euler's approximation.

Question 21 Answer (A)

The curves meet where $x^2 = 2x$, i.e. $x = 0$ and $x = 2$; on $(0, 2)$, $2x > x^2$. $\int_0^2 (2x - x^2)\,dx = \left[x^2 - \frac{x^3}{3}\right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$. Choices (B) and (C) are the areas under the individual curves; (D) adds them.

Question 22 Answer (D)

Disk method: $V = \pi \int_0^4 (\sqrt{x})^2\,dx = \pi \int_0^4 x\,dx = \pi\left[\frac{x^2}{2}\right]_0^4 = 8\pi$. Choice (C) forgets to square the radius ($\pi \int_0^4 \sqrt{x}\,dx$); (A) uses the y-range $0 \le y \le 2$ as the limits.

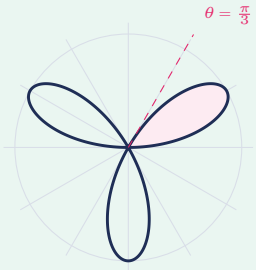
Question 23 Answer (C)

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t}$; at $t = 2$: $\frac{9}{4}$. Choice (A) inverts the quotient; (D) is dy/dt alone.

Question 24 Answer (A)

One petal is traced as θ runs from 0 to $\frac{\pi}{3}$ (where $\sin 3\theta = 0$ again).

$$A = \frac{1}{2}\int_0^{\pi/3} \sin^2(3\theta)\,d\theta = \frac{1}{2}\int_0^{\pi/3} \frac{1 - \cos(6\theta)}{2}\,d\theta = \frac{1}{4}\left[\theta - \frac{\sin 6\theta}{6}\right]_0^{\pi/3} = \frac{\pi}{12}.$$



$r = \sin(3\theta)$: the shaded petal corresponds to $0 \le \theta \le \frac{\pi}{3}$.

Choice (B) forgets the factor $\frac{1}{2}$ in $\frac{1}{2}\int r^2\,d\theta$.

Question 25 Answer (C)

Velocity $= \langle 2t,\ \pi\cos(\pi t)\rangle$; at $t = 1$ it is $\langle 2, -\pi\rangle$. Speed $= \sqrt{2^2 + (-\pi)^2} = \sqrt{4 + \pi^2}$. Choice (D) adds the component magnitudes instead of using the Pythagorean length.

Question 26 Answer (B)

Geometric series with first term $a = 3$ and ratio $r = -\frac{2}{5}$ ($|r| < 1$, so it converges): $S = \frac{a}{1 - r} = \frac{3}{1 + \frac{2}{5}} = \frac{15}{7}$. Choice (C) uses $r = +\frac{2}{5}$; (A) is the sum starting at $n = 1$.

Question 27 Answer (B)

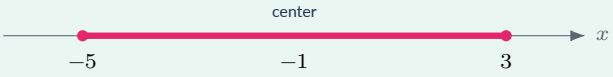
(B): $\frac{1}{\sqrt{n}}$ decreases to 0, so the alternating series converges by the Alternating Series Test, but $\sum \frac{1}{\sqrt{n}}$ is a divergent p-series ($p = \frac{1}{2} \le 1$). Hence (B) converges conditionally. (A) converges absolutely ($p = 2$). (C) diverges by the nth-term test, since $\frac{n}{n+1} \to 1 \neq 0$. (D) is a convergent p-series with positive terms, so its convergence is absolute.

Question 28 Answer (D)

$e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots$ with $u = -x^2$ gives $e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots$. Multiplying by x^2: $x^2 e^{-x^2} = x^2 - x^4 + \frac{x^6}{2} - \cdots$. The coefficient of x^6 is $\frac{1}{2}$. Choice (B) is the x^6 coefficient of e^{-x^2} itself (forgetting to multiply by x^2).

Question 29 Answer (A)

Ratio test: $\lim_{n\to\infty} \left| \frac{(x+1)^{n+1}}{(n+1)^2 4^{n+1}} \cdot \frac{n^2 4^n}{(x+1)^n} \right| = \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^2 \frac{|x+1|}{4} = \frac{|x+1|}{4}$, so the series converges for $|x+1| < 4$, i.e. $-5 < x < 3$. Endpoints: at $x = 3$ the series is $\sum \frac{1}{n^2}$, a p-series with $p = 2 > 1$, which converges; at $x = -5$ it is $\sum \frac{(-1)^n}{n^2}$, which converges absolutely (its absolute values form the same p-series). Interval: $-5 \le x \le 3$. Choice (B) skips the endpoint check; (C) treats $x = 3$ like the harmonic series (that would be the case for n rather than n^2 in the denominator); (D) centres the interval at $x = 1$ instead of $x = -1$.



Question 30 Answer (B)

$P_3(x) = 2 - 3(x-1) + \frac{4}{2!}(x-1)^2 + \frac{12}{3!}(x-1)^3$. With $x - 1 = 0.5$: $P_3(1.5) = 2 - 1.5 + 2(0.25) + 2(0.125) = 2 - 1.5 + 0.5 + 0.25 = 1.25$. Choice (A) stops at degree 2; (C) divides the cubic term by 2 instead of $3!$ (giving 0.75 in place of 0.25); (D) omits the factorials altogether.

Part B (calculator)

Question 31 Answer (D)

f changes sign on each of the intervals $[1,2]$, $[2,3]$, $[3,4]$, $[4,5]$ (signs $+, -, +, -, +$). By the Intermediate Value Theorem, f has at least one zero in each, so at least 4 zeros. Choice (A) applies the IVT only once to the whole interval.

Question 32 Answer (B)

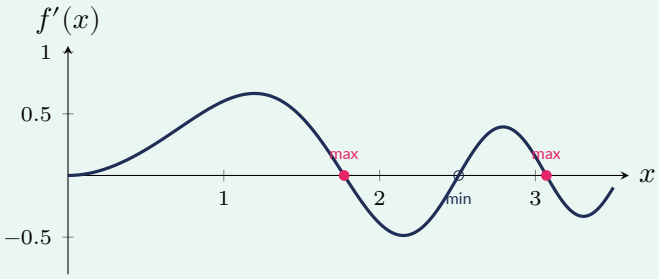
Parallel tangent lines require $f'(x) = g'(x)$: $\frac{1}{x+2} = 2x$, i.e. $2x^2 + 4x - 1 = 0$, so $x = \frac{-2 \pm \sqrt{6}}{2}$. Only $x = 0.224744\ldots \approx 0.225$ is in the domain $x > -2$ of f; the other root -2.225 (choice A) is extraneous. Choice (C) uses $g'(x) = x$; (D) is where the graphs *intersect*.

Question 33 Answer (B)

$a(t) = v'(t) = \sin\left(\frac{t^2}{2}\right) + t \cdot t \cos\left(\frac{t^2}{2}\right)$, so $a(2) = \sin 2 + 4\cos 2 = 0.909297 - 1.664587 = -0.755$. (Calculator: numerical derivative of v at $t = 2$ gives -0.75529.) Choice (A) keeps only the second product-rule term; (C) forgets the chain-rule factor t; (D) is $v(2)$.

Question 34 Answer (A)

$e^{-x/3} > 0$, so f' has the sign of $\sin(x^2)$, which is 0 when $x^2 = \pi, 2\pi, 3\pi$ (all less than $3.5^2 = 12.25 < 4\pi$): $x = \sqrt{\pi} = 1.772$, $\sqrt{2\pi} = 2.507$, $\sqrt{3\pi} = 3.070$. f' changes from positive to negative at $x = 1.772$ and $x = 3.070$ (relative maxima) and from negative to positive at $x = 2.507$ (relative minimum). A graph of f' on the calculator confirms the three sign changes.



Graph of $f'(x) = e^{-x/3}\sin(x^2)$ on $0 < x < 3.5$: f' crosses from $+$ to $-$ at the two relative maxima of f.

Question 35 Answer (D)

Amount at $t = 6 = 100 + \int_0^6 \left(20 + 10\sin\frac{t}{3}\right)dt = 100 + 162.484 = 262.484$ gallons. (Exactly: $100 + 120 + 30(1 - \cos 2)$.) Choice (B) omits the initial 100 gallons; (A) adds the *rate* $R(6)$ instead of the accumulated amount.

Question 36 Answer (B)

The subintervals have unequal widths $2, 3, 1, 4$:

$$2 \cdot \frac{3+7}{2} + 3 \cdot \frac{7+4}{2} + 1 \cdot \frac{4+8}{2} + 4 \cdot \frac{8+2}{2} = 10 + 16.5 + 6 + 20 = 52.5.$$

Choices (C) and (A) are the left and right Riemann sums (63 and 42); (D) forgets the factor $\frac{1}{2}$.

Question 37 Answer (A)

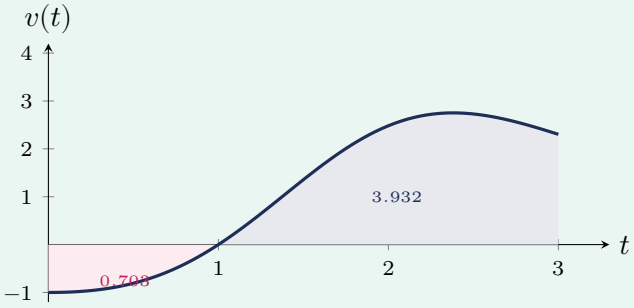
$y = 50e^{kt}$ and $80 = 50e^{3k}$ give $k = \frac{1}{3}\ln\frac{8}{5} = 0.156668$. Then $y(10) = 50\left(\frac{8}{5}\right)^{10/3} = 239.536$. Choice (B) uses $k = \frac{30/50}{3} = 0.2$ (a linear rate in place of the exponential constant); (C) starts the growth from 80 instead of 50, computing $80\left(\frac{8}{5}\right)^{10/3}$; (D) forgets to divide $\ln\frac{8}{5}$ by 3.

Question 38 Answer (C)

For a logistic model, $\frac{dP}{dt}$ is greatest when P is half the carrying capacity: $P = 600$. Solve $\frac{1200}{1 + 5e^{-0.3t}} = 600$: $5e^{-0.3t} = 1$, so $t = \frac{\ln 5}{0.3} = 5.365$. Choice (B) forgets to divide by 0.3; (D) comes from the algebra slip $e^{0.3t} = 1 + 5 = 6$.

Question 39 Answer (D)

Total distance $= \displaystyle\int_0^3 \left|(t-1)e^{\sin t}\right| dt = 4.635$ (calculator). Since $v < 0$ on $(0,1)$ and $v > 0$ on $(1,3)$, this equals $0.702713 + 3.931958$. Choice (B) is the displacement $\int_0^3 v(t)\,dt = 3.229$; (C) ignores the interval $0 \le t \le 1$; (A) is $v(3)$.



Distance adds both areas, $0.703 + 3.932 = 4.635$; displacement subtracts the area below the axis, $3.932 - 0.703 = 3.229$.

Question 40 Answer (D)

Intersections (calculator): $4 - x^2 = e^x$ at $x = a = -1.964636$ and $x = b = 1.058006$. On $[a,b]$, $4 - x^2 \ge e^x > 0$, so washers have outer radius $4 - x^2$ and inner radius e^x:

$$V = \pi \int_a^b \left[(4-x^2)^2 - (e^x)^2\right] dx = 84.705.$$

Choice (C) squares the difference, $\pi \int_a^b (4 - x^2 - e^x)^2 dx$ — the most common washer error; (A) is the area of R; (B) uses 0 as the lower limit.

Question 41 Answer (C)

Distance $= \displaystyle\int_0^2 \sqrt{\cos^2(t^2) + \left(e^{t/2}\right)^2}\, dt = \int_0^2 \sqrt{\cos^2(t^2) + e^t}\, dt = 3.828$. Choice (A) is the speed at $t = 2$; (B) is $\int_0^2 y'(t)\,dt$, the change in y only; (D) adds $|x'| + |y'|$ instead of using the speed.

Question 42 Answer (C)

$y = r\sin\theta = (2 + \sin 3\theta)\sin\theta$, so

$$\frac{dy}{d\theta} = 3\cos(3\theta)\sin\theta + (2 + \sin 3\theta)\cos\theta.$$

At $\theta = 1$: $3\cos 3\sin 1 + (2 + \sin 3)\cos 1 = -2.499\ldots + 1.157\ldots = -1.342$. Choice (A) is $\frac{dx}{d\theta}$, (B) is $\frac{dr}{d\theta}$, and (D) is $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$.

Question 43 Answer (B)

Lagrange error bound for the degree-4 polynomial: $\dfrac{\max|f^{(5)}|}{5!}|0.5 - 0|^5 = \dfrac{12}{120}\cdot\dfrac{1}{32} = \dfrac{1}{320}$. Choice (A) forgets the factor 12; (C) uses exponent and factorial 4 instead of 5; (D) omits the

$5!$.

Question 44 Answer (C)

Differentiate term by term (allowed inside the interval of convergence): $f'(x) = \sum_{n=1}^{\infty} \frac{n\,x^{n-1}}{n^2} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n}$. At $x = 0.5$: $\sum_{n=1}^{\infty} \frac{0.5^{\,n-1}}{n} = \frac{1}{0.5}\sum_{n=1}^{\infty} \frac{0.5^{\,n}}{n} = 2\ln 2 = 1.386$ (using $\sum \frac{x^n}{n} = -\ln(1-x)$; a calculator sum of the first 40 terms agrees to three decimals). Choice (A) is $f(0.5)$ itself; (B) is $\ln 2 = \sum \frac{0.5^n}{n}$, which drops the factor $\frac{1}{x}$.

Question 45 Answer (C)

The terms $\frac{1}{k^3}$ decrease to 0, so $|S - S_n| \le \frac{1}{(n+1)^3}$. We need $\frac{1}{(n+1)^3} < 0.0005$, i.e. $(n+1)^3 > 2000$. Since $12^3 = 1728$ and $13^3 = 2197$, the least choice is $n + 1 = 13$, so $n = 12$. Choice (D) bounds the error by the last term *used*, $\frac{1}{n^3}$, instead of the first term omitted.

11.8 Free-Response Solutions and Scoring

Each free-response question is worth 9 points. The scoring notes follow the style of College Board rubrics: a point is earned only when the stated work appears. "Bald" answers (a number with no supporting setup) earn no credit on parts that ask for justification or show work.

VN **GIẢI THÍCH TIẾNG VIỆT**

Cách tự chấm. Đọc từng dòng trong bảng điểm và hỏi: "bài mình có *viết ra* điều này không?" — nghĩ trong đầu mà không viết thì không tính. Một số quy ước quen thuộc của giám khảo AP: lỗi tính toán ở bước trước được "mang theo" (bài sau dùng số sai nhưng phương pháp đúng vẫn được điểm phần sau); câu hỏi có đơn vị thì thiếu đơn vị mất đúng 1 điểm đơn vị; khi trả lời câu có máy tính, phải ghi tích phân/phương trình rồi mới ghi kết quả 3 chữ số thập phân.

Question 1 (calculator) Solution

(a) $R'(6) \approx \dfrac{R(7) - R(5)}{7 - 5} = \dfrac{165 - 170}{2} = -2.5$ gallons per hour per hour.

(b)

$$\int_0^{12} R(t)\,dt \approx 2 \cdot \frac{130 + 150}{2} + 3 \cdot \frac{150 + 170}{2} + 2 \cdot \frac{170 + 165}{2} + 3 \cdot \frac{165 + 150}{2} + 2 \cdot \frac{150 + 120}{2}$$
$$= 280 + 480 + 335 + 472.5 + 270 = 1837.5.$$

$\int_0^{12} R(t)\,dt$ is the total number of gallons of water pumped into the tank during the 12 hours from $t = 0$ to $t = 12$.

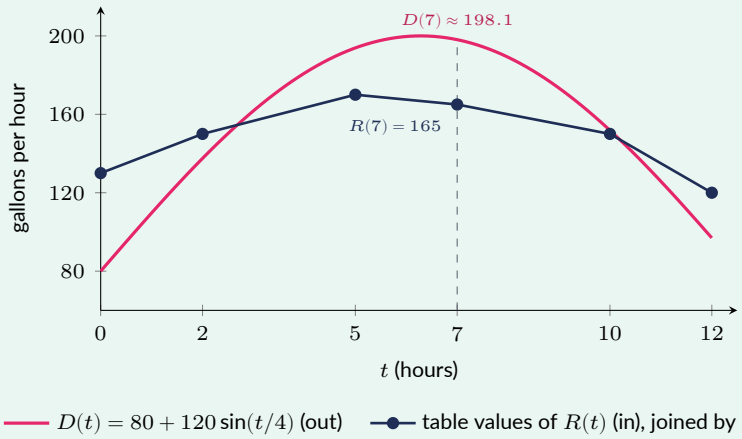
(c) R is differentiable, hence continuous, on $[2, 10]$, and $R(2) = 150 = R(10)$. By the Mean Value Theorem (Rolle's Theorem) there is a value c with $2 < c < 10$ such that $R'(c) = \dfrac{R(10) - R(2)}{10 - 2} = 0$. So yes, such a c must exist.

(d) Let $W(t)$ be the amount of water in the tank. Then

$$W(12) = 500 + \int_0^{12} R(t)\,dt - \int_0^{12} D(t)\,dt \approx 500 + 1837.5 - 1915.196 = 422.304 \text{ gallons.}$$

(Calculator: $\int_0^{12}\left(80+120\sin\frac{t}{4}\right)dt = 1915.19639\ldots$; exact value $960+480(1-\cos 3)$.)

(e) $W'(7) = R(7) - D(7) = 165 - \left(80+120\sin\frac{7}{4}\right) = 165 - 198.078 = -33.078 < 0$. The amount of water is **decreasing** at $t=7$ because water is drained faster than it is pumped in.



Rate in versus rate out. The segments joining the table values are exactly the tops of the trapezoids in part (b); at $t=7$ the drain rate exceeds the pump rate, so $W'(7)<0$.

Part	Pts	Scoring notes
(a)	1	Estimate -2.5 from a difference quotient of table values bracketing $t=6$, with units gal/hr^2 (the units point may be combined here).
(b)	2	1: trapezoidal sum with all five unequal widths; 1: value 1837.5 *and* meaning (total gallons pumped in over $0\le t\le 12$).
(c)	2	1: $R(2)=R(10)$; 1: answer "yes" with the MVT/Rolle's Theorem, stating that R is differentiable (so continuous).
(d)	3	1: $500+\int_0^{12}R\,dt-\int_0^{12}D\,dt$ (both integrals with the initial amount); 1: $\int_0^{12}D(t)\,dt=1915.196$; 1: answer 422.304 (using the part (b) value, with carried-forward credit).
(e)	1	Decreasing, with reason $R(7)<D(7)$ (or $W'(7)<0$).

Question 2 (calculator) Solution

(a) $x'(1.5)=1+2\sin(2.25)=2.556146$ and $y'(1.5)=e^{0.75}\cos(1.5)=0.149751$.

$$\text{Speed}=\sqrt{\left(x'(1.5)\right)^2+\left(y'(1.5)\right)^2}=2.561.$$

Acceleration: $x''(t)=4t\cos(t^2)$ and $y''(t)=e^{t/2}\left(\frac{1}{2}\cos t-\sin t\right)$, so $\mathbf{a}(1.5)=\langle x''(1.5),\,y''(1.5)\rangle=\langle -3.769,\,-2.037\rangle$.

(b) The particle moves to the left when $x'(t)<0$, i.e. $\sin(t^2)<-\frac{1}{2}$. For $0<t<3$ we have $0<t^2<9$, and $\sin(t^2)<-\frac{1}{2}$ exactly when $\frac{7\pi}{6}<t^2<\frac{11\pi}{6}$ (the next such interval starts at $t^2=\frac{19\pi}{6}>9$). Hence the particle moves left on

$$\sqrt{7\pi/6}<t<\sqrt{11\pi/6},\quad \text{i.e.}\quad 1.914<t<2.400,$$

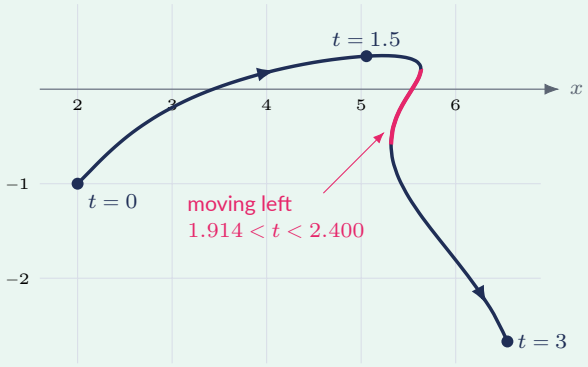
because $x'(t)<0$ there. (A graph of $x'(t)$ on $[0,3]$ shows the two zeros $t=1.914$ and $t=2.400$.)

(c)

$$x(3) = 2 + \int_0^3 (1 + 2\sin(t^2))\,dt = 2 + 4.547125 = 6.547,$$
$$y(3) = -1 + \int_0^3 e^{t/2}\cos t\,dt = -1 - 1.668771 = -2.669.$$

The particle is at $(6.547,\ -2.669)$.

(d) Distance $= \displaystyle\int_0^3 \sqrt{(1 + 2\sin(t^2))^2 + (e^{t/2}\cos t)^2}\,dt = 7.389.$



Path of the particle for $0 \le t \le 3$ (computed numerically from the velocity components). The pink arc is the time interval on which $x'(t) < 0$.

Part	Pts	Scoring notes
(a)	2	1: speed 2.561 (setup $\sqrt{x'^2 + y'^2}$ visible); 1: acceleration vector $\langle -3.769, -2.037\rangle$.
(b)	2	1: considers $x'(t) < 0$ as the condition for moving left; 1: interval $1.914 < t < 2.400$ with reason.
(c)	3	1: uses an initial condition with a definite integral ($2 + \int_0^3 x'\,dt$ or $-1 + \int_0^3 y'\,dt$); 1: $x(3) = 6.547$; 1: $y(3) = -2.669$.
(d)	2	1: integral $\int_0^3 \sqrt{x'(t)^2 + y'(t)^2}\,dt$; 1: answer 7.389.

Question 3 (no calculator) Solution

Areas between the graph of f and the x-axis: on $[-4, -2]$ a triangle of area 2 (above); on $[-2, 0]$ and on $[0, 2]$ quarter-discs of area $\frac{1}{4}\pi(2)^2 = \pi$ each (below); on $[2, 4]$ a triangle of area 2 (above); on $[4, 5]$ a triangle of area 1 (above, $f(5) = 0$); on $[5, 6]$ a triangle of area 1 (below).

(a)

$$g(-4) = \int_0^{-4} f(t)\,dt = -\int_{-4}^0 f(t)\,dt = -(2 - \pi) = \pi - 2,$$
$$g(6) = \int_0^6 f(t)\,dt = -\pi + 2 + 1 - 1 = 2 - \pi.$$

(b) $g'(x) = f(x)$. On $-4 < x < 6$, $f(x) = 0$ at $x = -2$, $x = 2$, and $x = 5$. $g'(x) = f(x)$ changes from positive to negative at $x = -2$ and at $x = 5$, so g has a relative maximum at $x = -2$ and at $x = 5$. (At $x = 2$, f changes from negative to positive: relative minimum.)

(c) Candidates: endpoints and the critical points $x = -2, 2, 5$.

x	-4	-2	2	5	6
$g(x)$	$\pi-2$	π	$-\pi$	$3-\pi$	$2-\pi$

Since $\pi > 3$, the values $3-\pi$ and $2-\pi$ lie between $-\pi$ and 0, and $\pi-2>0$. The absolute minimum value of g on $[-4,6]$ is $-\pi$, at $x=2$.

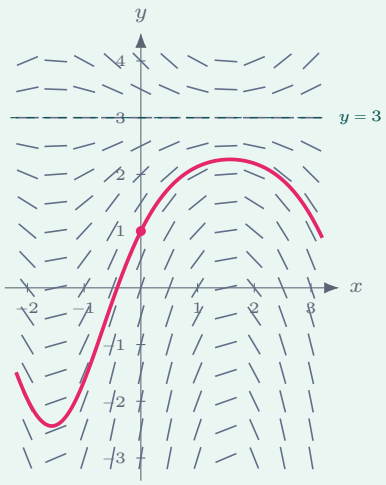
(d) $g(0)=0$ and $\sin 0=0$, and both functions are differentiable, so L'Hospital's Rule applies:

$$\lim_{x\to 0}\frac{g(x)}{\sin x}=\lim_{x\to 0}\frac{g'(x)}{\cos x}=\lim_{x\to 0}\frac{f(x)}{\cos x}=\frac{f(0)}{1}=-2.$$

Part	Pts	Scoring notes
(a)	2	1: $g(-4)=\pi-2$ (sign handled by reversing limits); 1: $g(6)=2-\pi$.
(b)	2	1: $x=-2$ and $x=5$; 1: justification "$g'=f$ changes from positive to negative".
(c)	3	1: considers $x=2$ (where g' changes from $-$ to $+$); 1: answer $-\pi$; 1: justification by comparing all candidate values (Candidates Test) or a complete global argument.
(d)	2	1: shows $\frac{0}{0}$ form ($g(0)=0$, $\sin 0=0$) and applies L'Hospital's Rule with $g'=f$; 1: answer -2.

Question 4 (no calculator) Solution

(a) The curve passes through $(0,1)$, follows the segments, stays below the equilibrium line $y=3$, and rises where $\cos x>0$ and falls where $\cos x<0$:



Solution through $(0,1)$; it is the graph of $y=3-2e^{-\sin x}$ found in part (d).

(b) $\left.\frac{dy}{dx}\right|_{(0,1)}=(3-1)\cos 0=2$. Tangent line: $y=1+2x$. Then $f(0.2)\approx 1+2(0.2)=1.4$.

(c) Differentiate $\frac{dy}{dx}=(3-y)\cos x$ with the product rule, remembering that y depends on x:

$$\frac{d^2y}{dx^2}=-\frac{dy}{dx}\cos x-(3-y)\sin x=-(3-y)\cos^2 x-(3-y)\sin x=-(3-y)(\cos^2 x+\sin x).$$

At $(0,1)$: $\frac{d^2y}{dx^2}=-2(1+0)=-2<0$. For $0\le x\le 0.2$ the solution stays below $y=3$ and $\cos^2 x+\sin x>0$, so $\frac{d^2y}{dx^2}<0$ there: the graph of f is concave down, it lies below its tangent line, and the approximation 1.4 is an **overestimate**.

(d) Separate the variables ($y \neq 3$):

$$\frac{dy}{3-y} = \cos x\,dx \implies -\ln|3-y| = \sin x + C.$$

Using $f(0)=1$: $-\ln 2 = 0+C$, so $C=-\ln 2$. Then $\ln|3-y| = \ln 2 - \sin x$, so $|3-y| = 2e^{-\sin x}$. Since $3-y=2>0$ at the initial point, $3-y=2e^{-\sin x}$ and

$$y = f(x) = 3 - 2e^{-\sin x} \qquad \text{for all real } x.$$

Check: $f'(x) = 2e^{-\sin x}\cos x = (3-f(x))\cos x$ and $f(0)=3-2=1$.

Part	Pts	Scoring notes
(a)	1	Solution curve through $(0,1)$, consistent with the slope field, extending in both directions and not crossing $y=3$.
(b)	1	Tangent line $y=1+2x$ and approximation $f(0.2)\approx 1.4$.
(c)	2	1: $\frac{d^2y}{dx^2} = -\frac{dy}{dx}\cos x - (3-y)\sin x$ (or equivalent); 1: "overestimate" with reason based on $\frac{d^2y}{dx^2}<0$ (concave down) near $x=0$.
(d)	5	1: separation of variables; 1: antiderivatives; 1: constant of integration; 1: uses initial condition; 1: solves for y: $y=3-2e^{-\sin x}$. (No credit for (d) without separation; max 2 of 5 if no constant of integration is used.)

Question 5 (no calculator) Solution

Intersection points. $3\cos\theta = 1+\cos\theta \iff \cos\theta = \frac{1}{2} \iff \theta = \pm\frac{\pi}{3}$. (The curves also share the pole, reached at $\theta=\frac{\pi}{2}$ by the circle and $\theta=\pi$ by the cardioid.) For $|\theta|\le\frac{\pi}{3}$ the circle is outside the cardioid.

(a)

$$\text{Area of } R = \frac{1}{2}\int_{-\pi/3}^{\pi/3}\Big[(3\cos\theta)^2 - (1+\cos\theta)^2\Big]d\theta = \int_0^{\pi/3}(8\cos^2\theta - 2\cos\theta - 1)\,d\theta.$$

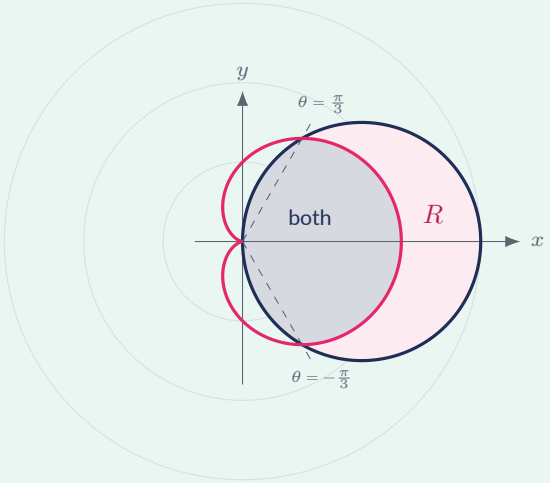
With $8\cos^2\theta = 4+4\cos 2\theta$:

$$\int_0^{\pi/3}(3+4\cos 2\theta - 2\cos\theta)\,d\theta = \Big[3\theta + 2\sin 2\theta - 2\sin\theta\Big]_0^{\pi/3} = \pi + 2\cdot\frac{\sqrt{3}}{2} - 2\cdot\frac{\sqrt{3}}{2} = \pi.$$

(b) The circle $r=3\cos\theta$ has diameter 3, so it encloses area $\pi\left(\frac{3}{2}\right)^2 = \frac{9\pi}{4}$. The region inside both curves is the disc minus R:

$$\frac{9\pi}{4} - \pi = \frac{5\pi}{4}.$$

(Equivalently, $2\left[\frac{1}{2}\int_0^{\pi/3}(1+\cos\theta)^2 d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2} 9\cos^2\theta\,d\theta\right] = \frac{5\pi}{4}$.)



Pink: region R (inside the circle, outside the cardioid), area π. Blue: region inside both curves, area $\frac{5\pi}{4}$.

(c) With $r = 1 + \cos\theta$: $x = r\cos\theta$, $y = r\sin\theta$, $\frac{dr}{d\theta} = -\sin\theta$.

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}.$$

At $\theta = \frac{\pi}{2}$: $r = 1$, $\frac{dr}{d\theta} = -1$, so $\frac{dy}{dx} = \frac{(-1)(1) + (1)(0)}{(-1)(0) - (1)(1)} = \frac{-1}{-1} = 1$.

(d) $\frac{dr}{dt} = \frac{dr}{d\theta}\cdot\frac{d\theta}{dt} = -\sin\theta\cdot 3$. At $\theta = \frac{\pi}{3}$: $\frac{dr}{dt} = -\frac{3\sqrt{3}}{2}$. Since $\frac{dr}{dt} < 0$, the particle's distance from the origin (the pole) is decreasing at a rate of $\frac{3\sqrt{3}}{2}$ units per unit of time at that instant.

Part	Pts	Scoring notes
(a)	3	1: integrand $\frac{1}{2}\left[(3\cos\theta)^2 - (1+\cos\theta)^2\right]$ (the constant $\frac{1}{2}$ and the difference of squares); 1: limits $\pm\frac{\pi}{3}$ (from solving $\cos\theta = \frac{1}{2}$); 1: answer π.
(b)	2	1: correct setup (disc area minus part (a), or a correct sum of polar integrals); 1: answer $\frac{5\pi}{4}$.
(c)	2	1: expression for $\frac{dy}{dx}$ in terms of θ (via $x = r\cos\theta$, $y = r\sin\theta$); 1: answer 1.
(d)	2	1: $\frac{dr}{dt} = -\frac{3\sqrt{3}}{2}$; 1: interpretation: distance from the origin is decreasing (at that rate).

Question 6 (no calculator) Solution

(a) Let $a_n = \dfrac{(-1)^{n+1}(x-2)^n}{n\cdot 3^n}$. Ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)3^{n+1}}\cdot\frac{n\,3^n}{(x-2)^n}\right| = \lim_{n\to\infty}\frac{n}{n+1}\cdot\frac{|x-2|}{3} = \frac{|x-2|}{3}.$$

The series converges when $\frac{|x-2|}{3} < 1$, i.e. $-1 < x < 5$ (radius 3).

- At $x = 5$: $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$, the alternating harmonic series; $\frac{1}{n}$ decreases to 0, so it converges by the Alternating Series Test.
- At $x = -1$: $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-3)^n}{n\,3^n} = \sum_{n=1}^{\infty}\frac{(-1)^{2n+1}}{n} = -\sum_{n=1}^{\infty}\frac{1}{n}$, which diverges (harmonic se-

ries).

The interval of convergence is $-1 < x \le 5$.

(b) Differentiate term by term:

$$f'(x) = \frac{1}{3} - \frac{x-2}{3^2} + \frac{(x-2)^2}{3^3} - \cdots + \frac{(-1)^{n+1}(x-2)^{n-1}}{3^n} + \cdots$$

(The first three nonzero terms are $\frac{1}{3} - \frac{x-2}{9} + \frac{(x-2)^2}{27}$. This is a geometric series with ratio $-\frac{x-2}{3}$, so $f'(x) = \frac{1}{x+1}$ for $-1 < x < 5$.)

(c) With $x = 3$, $x - 2 = 1$:

$$f(3) = \frac{1}{3} - \frac{1}{18} + \frac{1}{81} - \frac{1}{324} + \cdots \approx \frac{1}{3} - \frac{1}{18} = \frac{5}{18}.$$

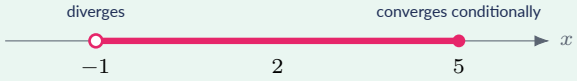
The series for $f(3)$ alternates, and the absolute values of its terms $\frac{1}{n\,3^n}$ decrease to 0. By the alternating series error bound,

$$\left| f(3) - \frac{5}{18} \right| \le \frac{1}{3 \cdot 3^3} = \frac{1}{81} < \frac{1}{50}.$$

(Indeed $f(x) = \ln \frac{x+1}{3}$, so $f(3) = \ln \frac{4}{3} \approx 0.2877$ and $\frac{5}{18} \approx 0.2778$: the error is about 0.0099.)

(d) Integrate term by term from 2 to x:

$$g(x) = \int_2^x f(t)\,dt = \frac{(x-2)^2}{2 \cdot 3} - \frac{(x-2)^3}{3 \cdot 2 \cdot 3^2} + \cdots = \frac{(x-2)^2}{6} - \frac{(x-2)^3}{54} + \cdots + \frac{(-1)^{n+1}(x-2)^{n+1}}{n(n+1)\,3^n} + \cdots$$



Interval of convergence of the series for f: $-1 < x \le 5$.

Part	Pts	Scoring notes
(a)	3	1: sets up the ratio $\left\lvert \frac{a_{n+1}}{a_n} \right\rvert$ and finds its limit $\frac{\lvert x-2 \rvert}{3}$; 1: identifies the open interval $-1 < x < 5$ (radius 3); 1: correct analysis of *both* endpoints with reasons, and the interval $-1 < x \le 5$.
(b)	2	1: first three nonzero terms $\frac{1}{3} - \frac{x-2}{9} + \frac{(x-2)^2}{27}$; 1: general term $\frac{(-1)^{n+1}(x-2)^{n-1}}{3^n}$.
(c)	2	1: approximation $\frac{1}{3} - \frac{1}{18} = \frac{5}{18}$; 1: error bound by the alternating series error bound, using the first omitted term $\frac{1}{81} < \frac{1}{50}$ (must state that the terms alternate and decrease in absolute value to 0).
(d)	2	1: first two nonzero terms $\frac{(x-2)^2}{6} - \frac{(x-2)^3}{54}$; 1: general term $\frac{(-1)^{n+1}(x-2)^{n+1}}{n(n+1)3^n}$.

11.9 Scoring Your Practice Exam

Section I and Section II each count for half of the exam. To combine them on the same scale, multiply the number of correct multiple-choice answers by 1.2 (so that 45 correct answers are worth 54 points, the same as the $6 \times 9 = 54$ free-response points):

$$\text{Composite} = 1.2 \times (\text{number correct in Section I}) + (\text{points in Section II}), \quad 0 \le \text{Composite} \le 108.$$

	Your score	Maximum
Section I: number correct (Q1–45)	_____	45
Section I weighted ($\times 1.2$)	_____	54
Section II: FRQ 1 + 2 + 3 + 4 + 5 + 6	_____	54
Composite score	_____	108

Composite score (out of 108)	Estimated AP score	Approximate share of points
63–108	5	58% and above
50–62	4	46%–57%
39–49	3	36%–45%
29–38	2	27%–35%
0–28	1	below 27%

8020 estimate, not an official College Board table. The College Board does not publish fixed cut scores; the conversion changes from year to year with the difficulty of each exam form. Use these ranges only to gauge your current level.

If you lost points on	Review
FRQ 1 (table, rate in/out)	Unit 6 Riemann and trapezoidal sums with unequal widths; Unit 5 Mean Value Theorem; Unit 8 accumulation $W(b) = W(a) + \int_a^b (\text{in} - \text{out})\, dt$.
FRQ 2 (planar motion)	Unit 9 speed $\sqrt{x'^2 + y'^2}$, acceleration vector, position from velocity with an initial condition, distance traveled.
FRQ 3 (graph of f, FTC)	Unit 6 accumulation functions $\int_a^x f(t)\, dt$ and areas from geometry; Unit 5 extrema, Candidates Test; Unit 4 L'Hospital's Rule.
FRQ 4 (differential equation)	Unit 7 slope fields, tangent-line approximation, $\frac{d^2y}{dx^2}$ by implicit differentiation, separation of variables.
FRQ 5 (polar)	Unit 9 polar area $\frac{1}{2}\int r^2\, d\theta$, area between polar curves, slope $\frac{dy}{dx}$ in polar form, meaning of $\frac{dr}{dt}$.
FRQ 6 (power series)	Unit 10 ratio test and endpoint analysis, term-by-term differentiation and integration, alternating series error bound.

VN **GIẢI THÍCH TIẾNG VIỆT**

Đọc kết quả thế nào? Bảng quy đổi trên là **ước lượng của 8020** dựa trên các năm trước, *không phải bảng chính thức* của College Board — mỗi năm ngưỡng điểm thay đổi theo độ khó của đề. Điều quan trọng hơn con số 1–5 là *bạn mất điểm ở đâu*: dùng bảng phân bổ Unit ở mục Answer Key để gom các câu sai theo Unit rồi quay lại chương tương ứng. Ở Section II, so riêng điểm từng câu: mất điểm ở câu Taylor (FRQ 6) hay câu cực (FRQ 5) thường do thiếu lập luận (điểm đầu mút, điều kiện của chuỗi đan dấu, hằng số $\frac{1}{2}$) hơn là do không biết cách làm. Làm lại các câu sai sau 3–5 ngày, không xem lời giải, để chắc rằng bạn đã thực sự nắm được.

Đồng hành cùng 8020 English

Cảm ơn bạn đã học cùng bộ giáo trình quốc tế 8020. **Điểm thật · Thầy thật · Kết quả thật** — nhiều học viên chỉ cần 1–2 khoá để đạt kết quả mong muốn.

Chương trình đào tạo tại 8020 English

IELTS Academic/General · **SAT** · PTE · Duolingo · TOEFL | **AP** (College Board) · **IB** Diploma · **Cambridge** IGCSE / A-Level | săn học bổng du học.

Vì sao chọn 8020 English?

- **100% giáo viên** đạt tối thiểu 8.0 IELTS hoặc 1500 SAT — đội ngũ Giáo sư · Tiến sĩ · Thạc sĩ tốt nghiệp trường top đầu thế giới (Carnegie Mellon — #1 Hoa Kỳ về Khoa học Máy tính).
- **Kèm 1–1** mọi độ tuổi; lớp sĩ số nhỏ 2–5 bạn (đa số dưới 10 bạn/lớp).
- Lộ trình cá nhân hoá, theo sát tiến độ từng học viên; lớp OFFLINE tại TP.HCM & ONLINE toàn quốc.

Bảng vàng học viên

AP 5/5 — Calculus AB/BC	**AP** 4–5 — Biology, Chemistry	**IB** 90/100 — Economics
SAT 1590 / 1570 / 1550	**IELTS** 8.0–9.0	**Duolingo** 110 · **PTE** 90

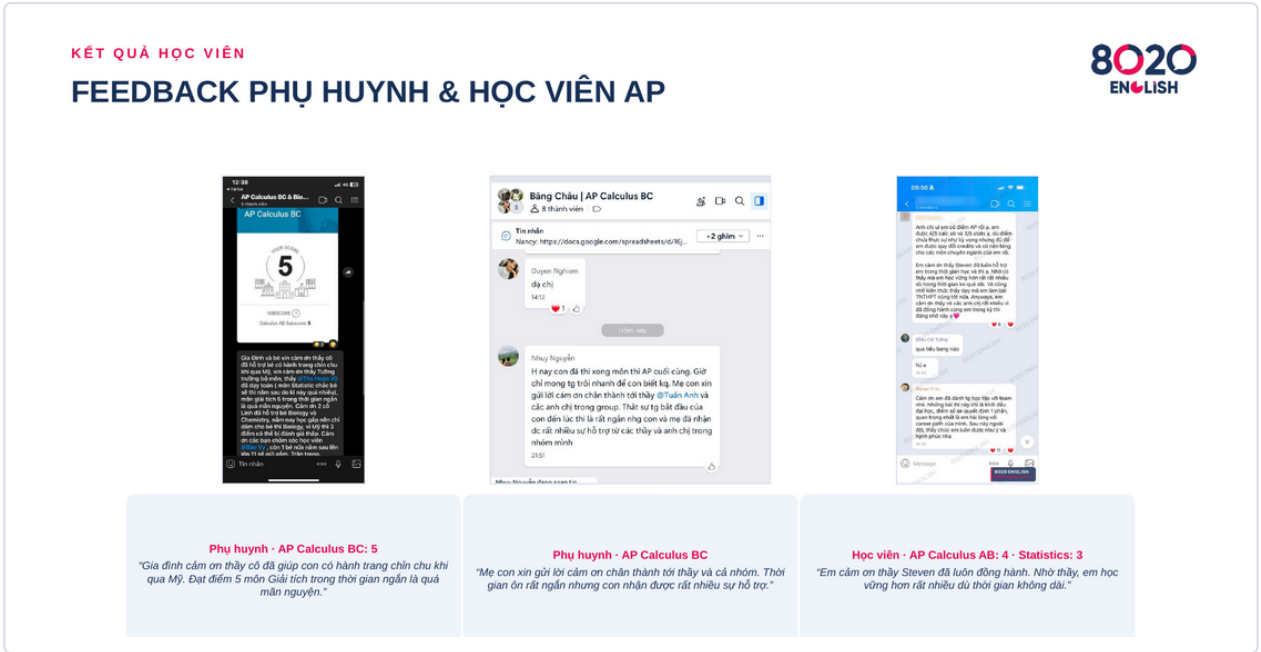
Cảm nhận học viên & phụ huynh

> *"Bạn đã cải thiện được tư duy Writing — kỹ năng từng là nỗi sợ lớn nhất — và cuối cùng đã đậu vào trường top như mong muốn. Thái độ học tập cực kỳ nghiêm túc; ngay cả khi nằm viện vẫn cố gắng học đều đặn."*
>
> — Phan Thị Thanh Hằng · IELTS Overall 8.5

> *"Cần tất cả kỹ năng trên 7.5 để xét vào trường top 1 Anh Quốc về Y học (chương trình UKFPO của NHS) — và đã đạt mục tiêu sau khi học Writing 1–1."*
>
> — Trần Hoàng Bảo Ngọc · IELTS Overall 8.0 (Writing 6.0 → 7.5)

> *"Gia đình và bé xin cảm ơn thầy cô đã giúp con có hành trang chỉn chu khi qua Mỹ. Đạt điểm 5 môn Giải tích trong thời gian ngắn là quá mãn nguyện."*
>
> — Phụ huynh · AP Calculus BC 5/5

> *"Em cảm ơn thầy đã luôn đồng hành. Nhờ thầy, em học vững hơn rất nhiều dù thời gian ôn không dài."*
>
> — Học viên · AP Calculus AB 4

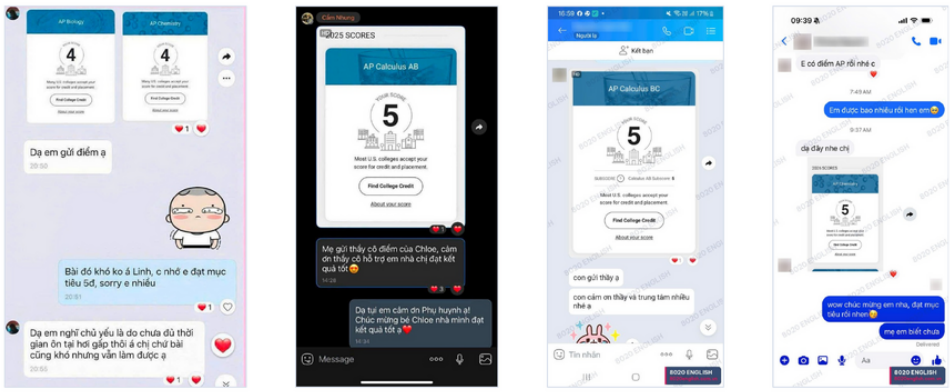


Feedback phụ huynh & học viên AP — nhiều môn đạt điểm 4 & 5

Điểm thi thật của học viên

KẾT QUẢ HỌC VIÊN

ĐIỂM SỐ AP



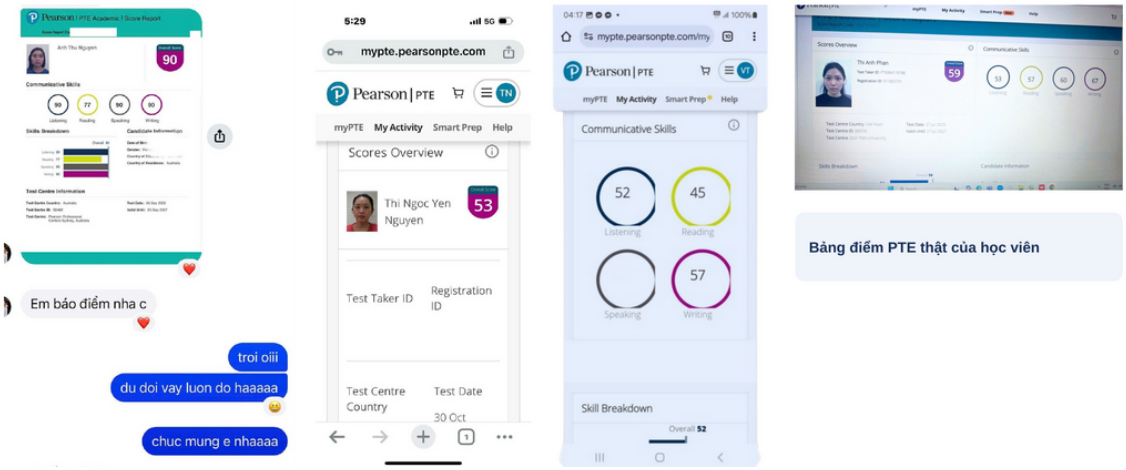
AP Biology: 4 · AP Chemistry: 4 | AP Calculus AB: 5 | AP Calculus BC: 5 | AP Chemistry: 5

Bảng điểm AP thật của học viên – nhiều môn đạt điểm 4 & 5

Bảng điểm AP thật — Calculus AB/BC 5/5 · Biology & Chemistry 4–5 (College Board)

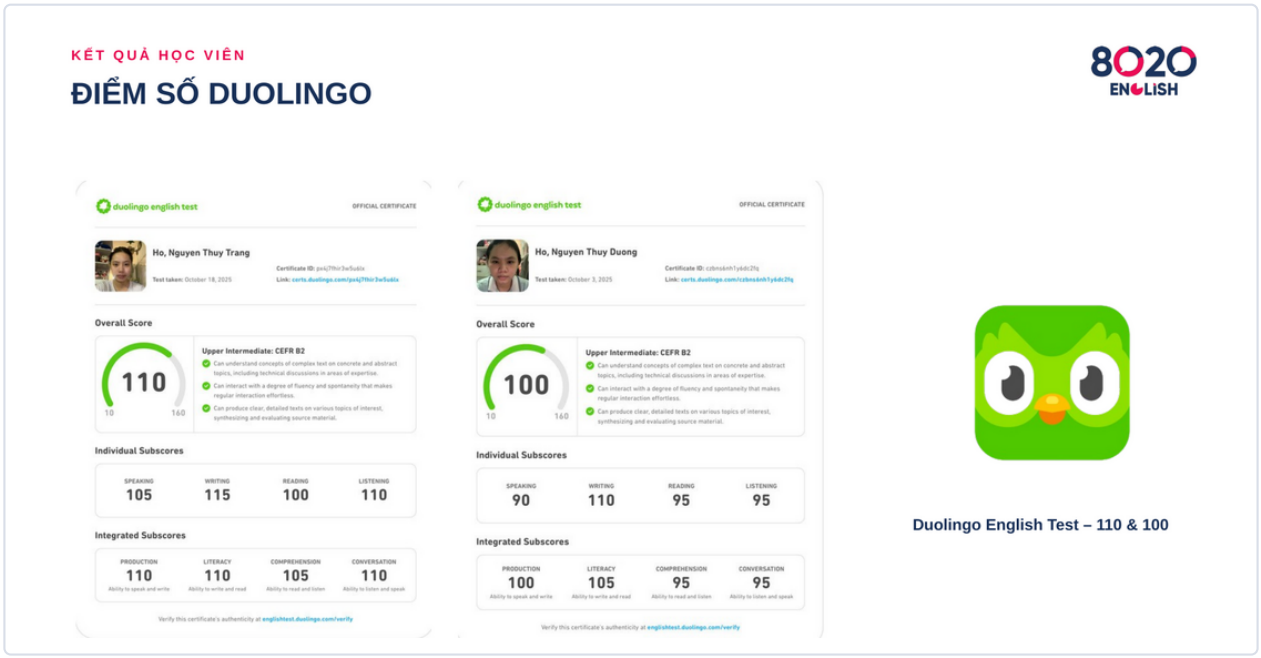
KẾT QUẢ HỌC VIÊN

ĐIỂM SỐ PTE

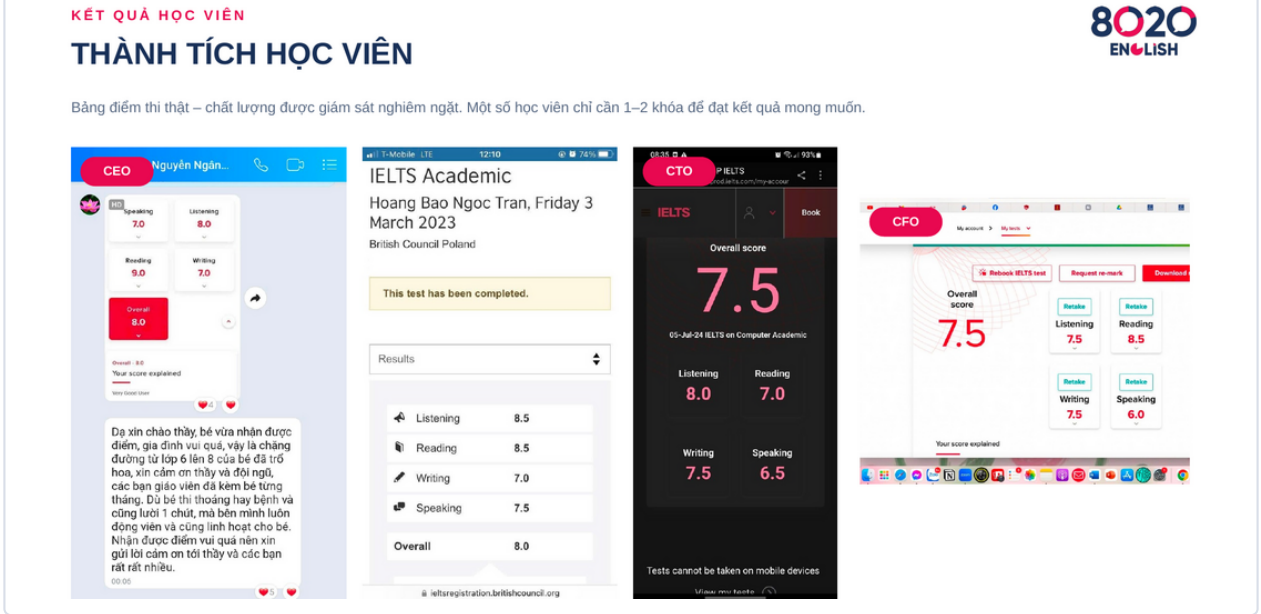


Bảng điểm PTE thật của học viên

Học viên 8020 — PTE Academic 90



Học viên 8020 — Duolingo English Test 110 & 100



Học viên 8020 — IELTS 8.0 / 7.5 / 8.5 (báo điểm thật)

8020 ENGLISH

8020 ENGLISH — CHƯƠNG TRÌNH QUỐC TẾ

Test đầu vào & tư vấn lộ trình MIỄN PHÍ

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Online	Lớp trực tuyến toàn quốc
Web	luyenthi8020.com

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